Name ________________________________

Student ID ____________________ TA/Section (circle):

Jimmy	B06-8am	B07-9am
	B08-10am	B01-4pm
Brandon	B02-5pm	B03-6pm
	B04-7pm	B05-8pm

Math 20B (Section B), Fall 2009, Midterm Exam 1 Solutions

- ***Show all of your work to receive full credit.***
- *Read each question carefully.*
- *Answer each question completely.*
- *Write your answers and work clearly and legibly; no credit will be given for illegible solutions.*
- *You are allowed one notebook-sized sheet of handwritten notes (written on one side).*
- *No calculators are allowed during the exam.*
- *50 minutes.*
- *Go back and check your answers if you finish early.*

Some trigonometric identities:

$$\sin(x+y) = \sin x \cos x + \cos x \sin y \qquad \cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\sin 2x = 2 \sin x \cos x \qquad \cos 2x = \cos^2 x - \sin^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

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1. (2 points) Let $f(x)$ be a continuous function. Express the following as a single integral

$$\int_5^3 f(x)\,dx + \int_3^8 f(x)\,dx = -\int_3^5 f(x)\,dx + \int_3^8 f(x)\,dx = \int_5^8 f(x)\,dx$$

$$\int_4^2 f(x)\,dx + \int_2^7 f(x)\,dx = -\int_2^4 f(x)\,dx + \int_2^7 f(x)\,dx = \int_4^7 f(x)\,dx$$

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2. (2 points) Calculate the derivative. (Justify your answer.)

$$\frac{d}{dx}\int_2^x \sin(2t^3)\,dt$$

By the Fundamental Theorem of Calculus, Part II, this is just $\sin(2x^3)$.

3. (2 points) Convert from polar to rectangular coordinates:

$$(r,\theta) = \left(-6, -\frac{3\pi}{2}\right) \qquad \text{OR} \qquad (r,\theta) = \left(-4, -\frac{3\pi}{2}\right)$$

Answer: The easiest way is to graph the point, then read off the coordinates. The $\theta = -\frac{3\pi}{2}$ direction is the same as $\theta = \frac{\pi}{2}$ which is straight up the positive y-axis. Since r is negative we go in the opposite direction, down the y-axis. For $(r,\theta) = \left(-6, -\frac{3\pi}{2}\right)$, this is $(x, y) = (0, -6)$. For $(r,\theta) = \left(-4, -\frac{3\pi}{2}\right)$, this is $(x, y) = (0, -4)$.
We can also use the formulas to compute

$$x = r\cos\theta = -6\cos\left(-\frac{3\pi}{2}\right) = -6(0) = 0$$
$$y = r\sin\theta = -6\sin\left(-\frac{3\pi}{2}\right) = -6(1) = -6$$

OR

$$x = r\cos\theta = -4\cos\left(-\frac{3\pi}{2}\right) = -4(0) = 0$$
$$y = r\sin\theta = -4\sin\left(-\frac{3\pi}{2}\right) = -4(1) = -4$$

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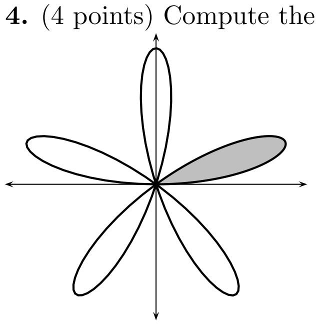
4. (4 points) Compute the area of one petal (shaded region) of the rose curve $r = \sin 5\theta$



Answer: The curve intersects the origin when $r = \sin 5\theta = 0$, which is when $5\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ which is when $\theta = 0, \pm\frac{\pi}{5}, \pm\frac{2\pi}{5}, \pm\frac{3\pi}{5}, \ldots$. The shaded region is contained in the area swept out by the curve between the first two intersections with the origin, which is between $\theta = 0$ and $\theta = \frac{\pi}{5}$. These become our limits of integration in the "area in polar coordinates" formula.

$$\frac{1}{2}\int_0^{\frac{\pi}{5}} r^2\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{5}} \sin^2 5\theta\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{5}} \frac{1}{2}\left(1 - \cos(10\theta)\right)\,d\theta = \frac{1}{4}\left[\theta - \frac{1}{10}\sin(10\theta)\right]_0^{\frac{\pi}{5}}$$
$$= \frac{1}{4}\left[\left(\frac{\pi}{5} - \frac{1}{10}\sin\left(10\frac{\pi}{5}\right)\right) - \left(0 - \frac{1}{10}\sin 0\right)\right] = \frac{\pi}{20}$$

5. (4 points) Compute the integral

$$\int xe^{-x}\,dx$$

Answer: Integration by parts:

$$u = x \qquad v' = e^{-x}$$
$$u' = 1 \qquad v = -e^{-x}$$

$$\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

Note that $\int e^{-x}\,dx$ is integrated using the substitution $u = -x$, $du = -dx$.

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6. (3 points) Compute the following and write your answer in the polar form $r(\cos\theta + i\sin\theta)$.

$$\left(\sqrt{2} - \sqrt{2}\,i\right)^5$$

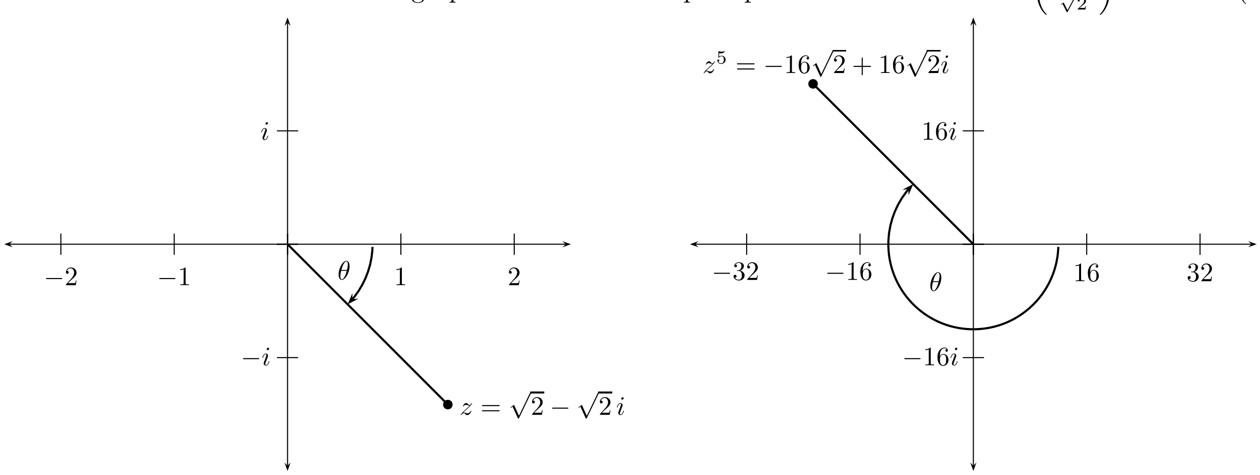
Answer: Let $z = \sqrt{2} - \sqrt{2}\,i$, then

$$r = |z| = \sqrt{\left(\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)^2} = \sqrt{4} = 2,$$

and

$$\theta = -\frac{\pi}{4} \qquad \text{or} \qquad \theta = \frac{7\pi}{4}.$$

The value of θ can be read from the graph of z in the complex plane or as $\theta = \arctan\left(\frac{-\sqrt{2}}{\sqrt{2}}\right) = \arctan(-1)$.



In polar form, $z = 2\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$. By de Moivre's Theorem, $z^5 = 2^5\left(\cos\left(-\frac{5\pi}{4}\right) + i\sin\left(-\frac{5\pi}{4}\right)\right)$.

If you used $\theta = \frac{7\pi}{4}$, then $z^5 = 2^5\left(\cos\left(\frac{35\pi}{4}\right) + i\sin\left(\frac{35\pi}{4}\right)\right)$.

In either case, it can be simplified to $z^5 = 32\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

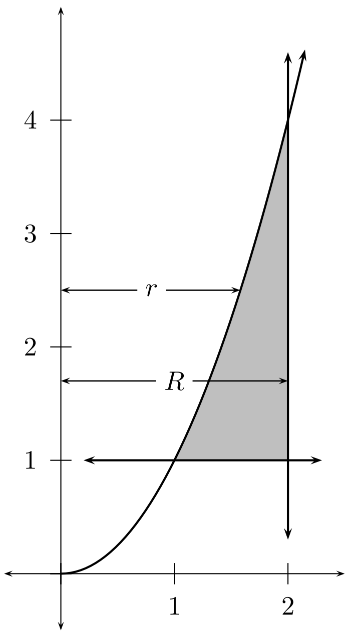
7. (4 points) A region is enclosed by the curve $y = x^2$, the line $y = 1$ and the line $x = 2$. Find the volume of the solid obtained by rotating this region about the y-axis.
Answer: We use the washer method, integrating along the y-axis

$$\text{Volume} = \pi \int_a^b (R^2 - r^2)\, dy.$$

Here, the inner radius is defined by the line $x = \sqrt{y}$, so $r = r(y) = \sqrt{y}$, and the outer radius is defined by the line $x = 2$, so $R = R(y) = 2$. This gives

$$\text{Volume} = \pi \int_1^4 (2^2 - \sqrt{x}^2)\, dy = \pi \int_1^4 (4 - y)\, dy = \pi \left[4y - \frac{1}{2}y^2\right]_1^4 = \pi \left[\left(4(4) - \frac{1}{2}(4)^2\right) - \left(4(1) - \frac{1}{2}(1)^2\right)\right]$$

$$= \pi \left[(16 - 8) - \left(4 - \frac{1}{2}\right)\right] = \pi \left[8 - \frac{7}{2}\right] = \frac{9}{2}\pi$$



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8. (4 points) Compute the integral

$$\int xe^{-x^2}\, dx$$

Answer: Do a substitution,

$$u = -x^2$$
$$du = -2x\, dx$$
$$-\frac{1}{2}\, du = x\, dx$$

to get

$$\int xe^{-x^2}\, dx = -\frac{1}{2} \int e^u\, du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$$