Name ____

Student ID ______________________ TA/Section (circle):

Jimmy	8am-B06	9am-B07
	10am-B08	4pm-B01
Brandon	5pm-B02	6pm-B03
	7pm-B04	8pm-B05

Math 20B (Section B), Fall 2009, Midterm Exam 2 Solutions

- ***Show all of your work to receive full credit.***
- *Read each question carefully.*
- *Answer each question completely.*
- *Write your answers and work clearly and legibly; no credit will be given for illegible solutions.*
- *You are allowed one notebook-sized sheet of handwritten notes (written on one side).*
- *No calculators are allowed during the exam.*
- *48 minutes.*
- *Go back and check your answers if you finish early.*

#	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
Σ	35	

1. (5 points) Determine convergence or divergence by any method. Justify your answer.

$$\sum_{n=1}^{\infty}\left(\frac{\ln n}{n}\right)^n$$

Solution: The power n leads us to think that the Root Test might work, so we look at

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{\ln n}{n}\right)^n\right|}=\lim_{n\to\infty}\frac{\ln n}{n}=0$$

We know this limit is zero by using L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{1/x}{1}=\lim_{x\to\infty}\frac{1}{x}=0$$

So by the Root Test, the series converges.

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2. (5 points) Evaluate the integral

Solution:

$$\int\frac{x^2}{(x-1)^2(x+1)}\,dx=\int\left[\frac{1/2}{(x-1)^2}+\frac{3/4}{x-1}+\frac{1/4}{x+1}\right]dx=-\frac{1}{2(x-1)}+\frac{3}{4}\ln|x-1|+\frac{1}{4}\ln|x+1|+C$$

Solution:

$$\int\frac{x}{(x-1)(x+1)^2}\,dx=\int\left[\frac{1/4}{x-1}+\frac{-1/4}{x+1}+\frac{1/2}{(x+1)^2}\right]dx=\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-\frac{1}{2(x+1)}$$

Solution:

$$\int\frac{x^2}{(x-1)(x+1)^2}\,dx=\int\left[\frac{1/4}{x-1}+\frac{3/4}{x+1}+\frac{-1/2}{(x+1)^2}\right]dx=\frac{1}{4}\ln|x-1|+\frac{3}{4}\ln|x+1|+\frac{1}{2(x+1)}$$

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3. (5 points) Determine the limit of the sequence $\{a_n\}$ or show that the sequence diverges as $n\to\infty$. Justify your answer.

$$a_n=\frac{\sin^2 n}{\sqrt{n}}$$

Solution: For $n\geq 1$ we have

$$0\leq\sin^2 n\leq 1$$

and

$$0\leq\frac{\sin^2 n}{\sqrt{n}}\leq\frac{1}{\sqrt{n}}$$

Since $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$, the Squeeze Theorem tells us that $\lim_{n\to\infty}\frac{\sin^2 n}{\sqrt{n}}=0$ also.

4. (5 points) Determine if the infinite sum is absolutely convergent, conditionally convergent, or divergent. Justify your answer.

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{2n+1}$$

Solution: The terms of the series do not converge to zero.

$$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+1/n} = \frac{1}{2}$$

so the $(-1)^n$ factor makes the even terms a_{2n} tend to $\frac{1}{2}$ and the odd terms a_{2n+1} tend to $-\frac{1}{2}$. Since the individual terms of the series do not tend to zero as $n \to \infty$, by the Divergence Test the series diverges.

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5. (5 points) Evaluate the integral

$$\int \cos^3 x\, dx$$

or

$$\int \sin^3 x\, dx$$

Solution:

$$\int \cos^3 x\, dx = \int (1 - \sin^2 x) \cos x\, dx$$

Let $u = \sin x$ and $du = \cos x\, dx$ to get

$$= \int \left[1 - u^2\right] du = u - \frac{1}{3}u^3 + C = \sin x - \frac{1}{3}\sin^3 x + C$$

Solution:

$$\int \sin^3 x\, dx = \int (1 - \cos^2 x) \sin x\, dx$$

Let $u = \cos x$ and $-du = \sin x\, dx$ to get

$$= \int (1 - u^2)\,(-du) = \int (u^2 - 1)\, du = \frac{1}{3}u^3 - u + C = \frac{1}{3}\cos^3 x - \cos x + C$$

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6. (5 points) Determine, with justification, whether the following integral converges or diverges.

$$\int_1^{\infty} \frac{dx}{x + e^x}$$

Solution: We cannot find an antiderivative of this integrand to evaluate the improper integral directly, but we can compare it to something that we know converges. For $x \geq 1$ we have

$$0 \leq \frac{1}{x + e^x} \leq \frac{1}{e^x} = e^{-x}$$

so we must have

$$\int_1^{\infty} \frac{dx}{x + e^x} \leq \int_1^{\infty} e^{-x}\, dx$$

Since

$$\int_1^{\infty} e^{-x}\, dx = \lim_{R\to\infty} \int_1^{R} e^{-x}\, dx = \lim_{R\to\infty} -e^{-x}\Big|_1^R = \lim_{R\to\infty} \left(-e^{-R} + e^{-1}\right) = e^{-1} = \frac{1}{e}$$

the larger integral converges, so by the Comparison Test for Improper Integrals, $\displaystyle\int_1^{\infty} \frac{dx}{x + e^x}$ converges.

7. (5 points) Determine convergence or divergence by any method. Justify your answer.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

Solution:

Since $a_n = f(n)$ where $f(x) = \frac{1}{x(\ln x)^2}$, and this looks like something we can integrate, we would like to apply the Integral Test to determine if this converges. First, $f(x)$ is continuous, positive and decreasing as x gets bigger. This is fairly obvious since both factors in the denominator are increasing as x gets bigger.

To be more exact we could say $\ln x$ is increasing and positive for $x > 1$, so $(\ln x)^2$ is also increasing, and x is increasing, so $x(\ln x)^2$ is increasing, so $\frac{1}{x(\ln x)^2}$ is decreasing.

Or we could say that

$$\frac{d}{dx}\left[x(\ln x)^2\right] = (\ln x)^2 + 2\ln x > 0$$

for $x > 1$, so the denominator is increasing for $x > 1$.

Now,

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{R\to\infty} \int_2^{R} \frac{dx}{x(\ln x)^2}$$

Let $u = \ln x$ and $du = \dfrac{dx}{x}$,

$$= \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{du}{u^2} = \lim_{R\to\infty} \left[-\frac{1}{u}\right]_{\ln 2}^{\ln R} = \lim_{R\to\infty} \left[-\frac{1}{\ln R} + \frac{1}{\ln 2}\right] = \frac{1}{\ln 2} < \infty$$

The integral converges, so by the Integral Test the series also converges.