Math 110A Midterm

Please answer the following questions. Because this test is open book, you will not get credit for answers unless you demonstrate how you arrived at them. In short, please show all work.

**Problem 1 (30 pts total)**

a) (15 pts.) Find the general solution to the transport equation:
\[ \partial_t u + x \partial_x u = 0, \quad u(x, 0) = u_0(x), \]
where \( u_0(x) \) is arbitrary.

b) (15 pts.) Find the specific solution to the transport equation:
\[ \partial_t u + \cos(t) \partial_x u - (x + t)u = 0, \quad u(x, 0) = x. \]

**Problem 2 (30 pts total)**

a) (15 pts.) Let \( u(t, x) \) be the solution to the wave equation \( u_{tt} = u_{xx} \) with initial data:
\[ u(x, 0) = 0, \quad u_t(x, 0) = \begin{cases} 1 + \cos(x), & |x| \leq \pi; \\ 0, & |x| > \pi. \end{cases} \]
Compute \( E(100) \) where \( E(t) = \frac{1}{2} \int (u_t^2 + u_x^2) dx \). (Recall \( \cos^2(x) = \frac{1}{2}(1 + \cos(2x)) \).)

b) (15 pts.) Let \( u(x, t) \) solve the wave equation \( u_{tt} = c^2 u_{xx} \). Show that the momentum:
\[ P(t) = \int_{-\infty}^{\infty} u_t u_x dx, \]
is constant in time. (Assume \( u_t, u_x \to 0 \) as \( |x| \to \infty \).)

**Problem 3 (20 pts total)**

a) (10 pts) Let \( u(x, t) \) be the solution to the wave equation with reflecting boundary condition at \( x = 0 \):
\[ u_{tt} = 4u_{xx}, \quad u(0, t) = 0, \quad u(x, 0) = 0, \quad u_t(x, 0) = \begin{cases} 1 - |x - 1| , & |x - 1| \leq 1; \\ 0, & |x - 1| > 1. \end{cases} \]
Compute the value \( u(\frac{1}{2}, \frac{3}{4}) \).

b) (10 pts) Show that \( u(x, t) = 0 \) whenever \( 0 \leq x \leq 2t - 2 \).

**Problem 4 (20 pts total)**

Let \( u(x, t) \) solve the inhomogeneous wave equation:
\[ u_{tt} - u_{xx} = F, \quad u(x, 0) = u_t(x, 0) = 0, \]
where \( F \) is the function:
\[ F(x, t) = \begin{cases} 1, & \text{if } -1 \leq x \leq 1 \text{ and } 0 \leq t \leq 1; \\ 0, & \text{otherwise}. \end{cases} \]
a) (10 pts) Compute the value of \( u(x, t) \) in the region \( 0 \leq |x| \leq t - 2 \).

b) (10 pts) Compute the value of \( u(x, t) \) in the region \( 1 \leq t + 1 \leq x \).