Please answer the following questions. Because this test is open book and open note, you will not get credit for answers unless you demonstrate how you arrived at them. In short, please show all work.

PROBLEM 1. (50 pts)

In this question we are considering the following mixed boundary value problem for the heat equation:

\begin{align}
(1a) & \quad u_t = ku_{xx} , \quad x \in [0, \ell] , \\
(1b) & \quad u(t, 0) = u_x(t, \ell) = 0 , \\
(1c) & \quad u(0, x) = f(x) .
\end{align}

Please answer the following:

a) First, find a set of elementary solutions of the form $u(t, x) = T(t)X(x)$ by finding all solutions to the problem:

\begin{align}
- X''_n &= \lambda_n X_n , \quad x \in [0, \ell] , \\
X_n(0) &= X'_n(\ell) .
\end{align}

Please justify your answer with your own computations.

b) Show that the following holds for the $X_n$ you found in part a) without using any trigonometric identities:

\[ \int_0^\ell X_n(x)X_m(x)dx = 0 , \quad n \neq m . \]
Problem 1 cont.

c) Let \( f(x) = x^2 - 2x \) be the initial data for the system (1) above where we choose \( \ell = 1 \). Compute the explicit solution \( u(t, x) \) in this case. (Hint: That is, first expand \( f(x) = \sum_n A_n X_n \) and then compute the solution to the heat equation from this.)

c) For the solution from part c) above, find the steady-state temperature \( T_\infty = \lim_{t \to \infty} u(t, x) \). Why does this make sense from a physical perspective?
Problem 2. (30 pts)

In this problem we are considering solutions to the following Dirichlet BC wave equation:

\( u_{tt} = c^2 u_{xx}, \quad x \in [0, 2], \)

(2a)

\( u(t, 0) = u(t, 2) = 0, \)

(2b)

\( u(0, x) = f(x), \)

(2c)

\( u_t(0, x) = g(x). \)

(2d)

Please answer the following:

a) In a class lecture we showed that there is a series expansion:

\[
2x - x^2 = \sum_{n \geq 0} 4 \left( \frac{2}{(2n + 1)\pi} \right)^3 \sin \left( \frac{(2n + 1)\pi}{2} x \right).
\]

Using this, compute the solution \( u(t, x) \) to the system (2) with initial data \( f(x) = 0 \) and \( g(x) = 2x - x^2. \)

b) Compute the energy:

\[
E(t) = \frac{1}{2} \int_0^2 (u_t^2 + u_x^2) dx,
\]

at time \( t = 10 \) for the solution \( u(t, x) \) you found from part a) above.
Problem 3. (20 pts)

For this problem we set \( f(x) \) to be the following discontinuous periodic function defined on \( x \in [-\pi, \pi] \):

\[
f(x) = \begin{cases} 
-1, & -\pi \leq x \leq 0; \\
1, & 0 < x < \pi.
\end{cases}
\]

Please answer the following:

a) Compute the full sin/cosine series of \( f(x) \). (Hint: It goes faster if you notice that \( f(x) \) is essentially odd.)

b) Let \( f_N = \frac{1}{2} A_0 + \sum_{n=1}^{N} (A_n \cos(nx) + B_n \sin(nx)) \) be the partial fourier series for \( f(x) \). Show that \( f_N \to f \) in the mean square sense.

c) (10 pts. Extra Credit) Does \( f_N \to f \) uniformly? Try to justify your answer as best you can.