Please answer the following questions. You will not get partial credit for answers unless you demonstrate how you arrived at them. In short, please show all work.

**Problem 1.**

Please solve the following transport equation:

\[ xu_x + 2u_y = 0 , \]

with the initial conditions \( u(x, 0) = \sin(x) ). \)
Problem 2.

In this problem we are considering the following initial value problem for the wave equation \(\text{without any boundary conditions}:\)

\[
\begin{align*}
  u_{tt} - u_{xx} &= 0 , \\
  u(0, x) &= \frac{3}{1 + x^2} , \\
  u_t(0, x) &= \frac{-2x}{(1 + x^2)^2} .
\end{align*}
\]

(a) Use D’Alembert’s formula to write out the explicit solution \(u(t, x)\) to this problem.

(b) Show that for every fixed \(x\), one has the limit \(\lim_{t \to \infty} u(t, x) = 0\). Notice that this is consistent with \(\int_{-\infty}^{\infty} u_t(0, x) \, dx = 0\).
Problem 3.

The purpose of this problem is to understand how to solve the wave equation with the following mixed boundary conditions:

\[(1a) \quad u_{tt} = c^2 u_{xx}, \quad x \in [0, \ell],\]
\[(1b) \quad u_x(t, 0) = 0,\]
\[(1c) \quad u(t, \ell) = 0,\]
\[(1d) \quad u(0, x) = f(x),\]
\[(1e) \quad u_t(0, x) = g(x).\]

a) Find all solutions to the separation of variables problem:
\[-X_n'' = \lambda_n X_n, \quad x \in [0, \ell],\]
\[X_n'(0) = X_n(\ell) = 0.\]

b) Use the answer from part a) above to find the explicit solution to (1) with \(\ell = \pi\) initial data:
\[f(x) = x \sin(x) \quad g(x) = x^2 - \pi^2.\]

(Hint: To compute the Fourier coefficients of \(f(x)\) use double angle identities and then integration by parts.)
Problem 4.

The purpose of this problem is to understand how to solve the heat equation with \textit{inhomogeneous} boundary conditions:

\begin{align*}
(2a) & \quad u_t = ku_{xx} , \quad x \in [0, \ell] , \\
(2b) & \quad u(t, 0) = a , \\
(2c) & \quad u(t, \ell) = b , \\
(2d) & \quad u(0, x) = f(x) .
\end{align*}

Here we may have $a \neq 0$ or $b \neq 0$. Please answer the following:

a) Show that the boundary conditions (2b)–(2c) are not symmetric unless $a = b = 0$.

b) First solve the equation (2a) with boundary conditions (2b)–(2c) in the case where $u_t = 0$. This is the \textit{steady state} solution.

c) In the general case, let $\tilde{u} = u - u_0$ where $u_0$ is the steady state with boundary conditions (2b)–(2c). Show that $\tilde{u}$ solves the Dirichlet problem:

\begin{align*}
(3a) & \quad \tilde{u}_t = k\tilde{u}_{xx} , \quad x \in [0, \ell] , \\
(3b) & \quad \tilde{u}(t, 0) = 0 , \\
(3c) & \quad \tilde{u}(t, \ell) = 0 , \\
(3d) & \quad \tilde{u}(0, x) = f(x) - u_0(x) .
\end{align*}
d) Use the above process to solve the system (2) in the case where $\ell = 1$, $a = 2$, $b = -1$, and the initial data is the function $f(x) = -3x^2 + 2$ in terms of an explicit series. Compute the limit $u_0(x) = \lim_{t \to \infty} u(t, x)$ in this case.
Problem 6.

In this problem, you will solve the 2D Laplace equation with Neumann boundary conditions:

(a) Suppose that we first expand:

\[ h(\theta) = \frac{1}{2} A_0 + \sum_{n \geq 1} \left( A_n \cos(n\theta) + B_n \sin(n\theta) \right). \]

Write out a formula for the solution \( u(r, \theta) \) to (4) in terms of a series involving \( A_n \) and \( B_n \).

(b) Use the formula you found in part a) above to show that one must have the following condition on \( h(\theta) \) from (4b):

\[ \int_0^{2\pi} h(\theta) d\theta = 0. \]

Can you come up with a physical reason why this should make sense, e.g. in terms of steady state temperature distributions?

(c) Write down the explicit solution \( u(r, \theta) \) to the problem (4) with \( a = 10 \) and:

\[ h(\theta) = 5 \sin(\theta) - 8 \cos(7\theta) + \sin(11\theta). \]

Is this solution unique? If not, find all possible solutions.