Consider the matrix equation:

\[ \dot{x} = Ax, \quad \text{where} \quad A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -1 \\ 3 & -1 & 0 \end{bmatrix}. \]

It turns out that the characteristic polynomial for this matrix is \( p_A(\lambda) = (\lambda + 1)(\lambda - 1)(\lambda - 2) \). Use this information to find the specific solution with \( x(0) = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \).

Consider the matrix equation:

\[ \dot{x} = Ax , \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ -4 & 2 & 2 \\ -2 & 1 & 2 \end{bmatrix}. \]

It turns out that both \( \lambda = 2 \) and \( \mu = 1 + i \) are eigenvalues for \( A \). Use this information to find the specific solution with \( x(0) = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \).

Let \( A \in \mathcal{M}(4 \times 4) \) be a real matrix with complex eigenvalue/eigenvector pairs:

\[ \mu_1 = 2 - i, \quad z_1 = \begin{bmatrix} 1 \\ 1 + i \\ 2 \\ 3 + 2i \end{bmatrix}, \quad \mu_2 = 1 + i, \quad z_2 = \begin{bmatrix} i \\ 1 \\ 2 - i \\ 1 \end{bmatrix}. \]

Use this information to write the general solution to the problem:

\[ \dot{x} = Ax, \quad x(0) = a \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}. \]

Consider the following system of second order ODE for \( x, y \):

\[ x'' = -\partial_x f(x, y), \quad y'' = -\partial_y f(x, y), \quad \text{where} \quad f(x, y) = ax^2 + bxy + cy^2, \]

where \( a, b, c \in \mathbb{R} \) are constants.

a) Let \( (x(t), y(t)) \) be a solution to this system. Show that \( E(t) = \frac{1}{2}(x'^2(t) + y'^2(t)) + f(x, y) \) is a constant.

b) Write the system for \( (x, y) \) as a first order 4 \times 4 system, and compute its characteristic polynomial.
c) Suppose that $a, c > 0$ and $4ac > b^2$. Show in this case the eigenvalues for the system from b) above are purely imaginary (i.e. have zero real part). Explain why this is consistent with the conservation of energy from part a).