MATH 142A HOMEWORK 1

**Important:** Please answer each of these questions on a separate sheet(s) of paper. Also, put your name and section number on each sheet. You will then upload your final solutions GradeScope as explained on the class webpage.

**Problem # 1**

Use mathematical induction to prove each of the following claims:

a) The number $2^{2n+1} + 1$ is divisible by 3 for every $n \in \mathbb{N}$.

b) That $\sum_{k=1}^{n} k! = (n+1)! - 1$ for each $n \in \mathbb{N}$.

c) Let $f : [a, b] \to \mathbb{R}$ be a function such that $f(x + y)^n \leq f(x)^n + f(y)^n$ for all $x, y \in [a, b]$. Show that this implies:

$$f(x_1 + x_2 + \ldots + x_n) \leq \frac{1}{n} \left( f(x_1) + f(x_2) + \ldots + f(x_n) \right),$$

for each $n \in \mathbb{N}$ and collection $x_i \in [a, b]$. (Hint: First use induction to show it for $n = 2^m$ for $m = 1, 2, \ldots$ and then work backwards to fill in the gaps.)

**Problem # 2**

These problems concern the interplay of rational and irrational numbers.

a) Show that both $\sqrt[3]{3} + \sqrt{5}$ and $\sqrt[3]{3} - \sqrt{5}$ are irrational numbers.

b) Prove that for every $n, m \in \mathbb{N}$ one has the lower bound $|\sqrt{2} - \frac{m}{n}| \geq \frac{1}{4n^2}$. This shows there are some surprising limitations as to how well one can approximate $\sqrt{2}$ by rational numbers with denominators which are not too large. (Hints: Use algebra here and not induction. While you have to give a proof, you will find a calculator very useful for part of this problem.)

**Problem # 3**

These problems concern the notion of SUP and INF.

a) Let $S \subseteq \mathbb{R}$ be a bounded collection of real numbers. Define $abs(S) = \{x \in \mathbb{R} \mid x = |s| \text{ for some } s \in S\}$. In other words $abs(S)$ consists of the set of absolute values of elements of $S$. Prove that:

$$SUP(abs(S)) = max\{SUP(S), -INF(S)\}.$$

b) Using the same setup as in the previous problem find a simple formula for $INF(abs(S))$ and prove its validity.

c) Let $S \subseteq \mathbb{R}$ be a collection of real numbers (not necessarily bounded) such that $INF(S) = c > 0$. Prove that the set $\frac{1}{S} = \{x \in \mathbb{R} \mid x = \frac{1}{s} \text{ for some } s \in S\}$ is bounded and $SUP(\frac{1}{S}) = \frac{1}{c}$.

d) Using the same setup as in the previous problem, prove that $S$ is unbounded iff $INF(\frac{1}{S}) = 0$.

**Problem #4 (Construction of real numbers part I)**

A subset $\mathcal{K} \subseteq \mathbb{Q}$ is called a “cut” if it satisfies the following two properties:

1. $\mathcal{K}$ is bounded from above but has no greatest element.
2. If $x \in \mathcal{K}$ then $y \in \mathcal{K}$ for all $y < x$.

One can use cuts to construct real numbers directly from $\mathbb{Q}$. To get a sense of this prove the following:

a) Show that for each real number $r \in \mathbb{R}$ the set $\mathcal{K}_r = \{q \in \mathbb{Q} \mid q < r\}$ is a cut.

b) Show that the map $r \mapsto \mathcal{K}_r$ is a bijection between the set of real numbers $\mathbb{R}$ and the set of all cuts.
c) Define the sum of two cuts as follows:

\[ K_1 + K_2 = \{ p + q \mid p \in K_1 \text{ and } q \in K_2 \} . \]

Show that this operation is indeed well defined (in other words \( K_1 + K_2 \) is a cut), and that \( K_{r_1} + K_{r_2} = K_{r_1+r_2} \) for any two \( r_1, r_2 \in \mathbb{R} \).

d) Prove that the irrational number \( \sqrt{2} \) is defined by the cut:

\[ K = \{ q \in \mathbb{Q} \mid q^2 < 2 \} \cup \{ q \in \mathbb{Q} \mid q < 0 \} . \]

(Note: One can also define multiplication of cuts, but its a lot more tedious because of negative numbers. Try it for yourself if you are interested.)