**Important:** Please answer each of these questions on a separate sheet(s) of paper. Also, put your name and section number on each sheet. You will then upload your final solutions GradeScope as explained on the class webpage.

**Problem # 1**

Use mathematical induction to prove each of the following claims:

a) That \( \sum_{k=1}^{n} (2k-1)^2 = \frac{n(4n^2-1)}{3} \) for each \( n \in \mathbb{N} \).

b) Show that for each \( n \in \mathbb{N} \) and real \( r > -1 \) one has the identity:

\[
\int_{0}^{1} x^n (1-x)^r \, dx = \frac{n!}{(r+1) \cdot (r+2) \cdot \ldots \cdot (r+n+1)}. \]

(Its OK the use calculus here!)

c) Show that if \( a_1, \ldots, a_n > 0 \) are all real numbers then:

\[
\prod_{i=1}^{n} a_i = 1, \quad \text{implies} \quad \sum_{i=1}^{n} a_i \geq n.
\]

(Hints: First try to prove this using induction when \( n = 2^k \) for \( k \in \mathbb{N} \). Instead of the original statement try to show the modified version \( \prod_{i=1}^{2^k} a_i \leq 2^{-k} \sum a_i^2 \). You will find the inequality \( ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2 \) useful. Alternatively you can use induction directly by showing \( a + b \geq ab + 1 \) whenever \( a \geq 1 \geq b \).)

**Problem # 2**

These problems concern the interplay of rational and irrational numbers.

a) Show that both \( \sqrt[3]{2} - 1 \) and \( (1 + \sqrt{2})^\frac{3}{2} \) are irrational numbers.

b) Let \( \beta \in \mathbb{R} \) be an irrational number. Show that for every \( \epsilon > 0 \) one can find \( m, n \in \mathbb{Z} \) with \( n \neq 0 \) such that \( |m + n\beta| < \epsilon \). Note: For this problem it is OK to assume that all numbers of the form \( m + n\beta \) are contained in an ordered field \( \mathbb{R} \supset \mathbb{Q} \) with the property that for every \( x \in \mathbb{R} \) there exists \( n \in \mathbb{Z} \) with \( n \leq x < n + 1 \). (Hint: For each \( k \in \mathbb{N} \) consider the distance from \( n\beta \) to the nearest integer for \( n = 1, \ldots, k \). Use the pigeonhole principle to argue that least two of these numbers must be within \( \frac{1}{k} \) of each other.)

**Problem #3**

For this problem let \( \langle \mathbb{F}, < \rangle \) be an arbitrary ordered field.

a) Show for any two numbers \( a, b \in \mathbb{F} \) that one has \( |a| - |b| \leq |a-b| \).

b) Prove for any finite collection of numbers \( a_i \in \mathbb{F} \) that one has:

\[
\left| \sum_{i=1}^{n} a_i \right| \leq \sum_{i=1}^{n} |a_i|.
\]