**MATH 142A HOMEWORK 2**

**Important:** Please answer each of these questions on a separate sheet(s) of paper. Also, put your name and section number on each sheet. You will then upload your final solutions GradeScope as explained on the class webpage.

**Problem # 1**

These problems concern the notion of SUP and INF.

a) Compute both the INF and the SUP of the following sets (this could be ±∞):

\[ A = \left\{ \frac{n}{1 + m} \mid \text{where} \ m, n \in \mathbb{N} \right\}, \quad B = \left\{ \frac{n}{m} \mid \text{where} \ m, n \in \mathbb{N} \text{ and } n^2 < 2m^2 \right\}. \]

b) Let \( S \subseteq \mathbb{R} \) be a collection of real numbers (not necessarily bounded) such that \( INF(S) = c > 0 \). Prove that the set \( \frac{1}{S} = \{ x \in \mathbb{R} \mid x = \frac{1}{s} \text{ for some } s \in S \} \) is bounded and \( SUP(\frac{1}{S}) = \frac{1}{c} \).

**Problem #2 (Construction of real numbers part I)**

A subset \( K \subseteq \mathbb{Q} \) is called a “cut” if it satisfies the following two properties:

1. \( K \) is bounded from above but has no greatest element.
2. If \( x \in K \) then \( y \in K \) for all \( y < x \).

One can use cuts to construct real numbers directly from \( \mathbb{Q} \). To get a sense of this prove the following:

a) Show that for each real number \( r \in \mathbb{R} \) the set \( K_r = \{ q \in \mathbb{Q} \mid q < r \} \) is a cut.

b) Show that the map \( r \mapsto K_r \) is a bijection between the set of real numbers \( \mathbb{R} \) and the set of all cuts.

c) Define the sum of two cuts as follows:

\[ K_1 + K_2 = \{ p + q \mid p \in K_1 \text{ and } q \in K_2 \}. \]

Show that this operation is indeed well defined (in other words \( K_1 + K_2 \) is a cut), and that \( K_{r_1} + K_{r_2} = K_{r_1 + r_2} \) for any two \( r_1, r_2 \in \mathbb{R} \).

d) Prove that the irrational number \( \sqrt{2} \) is defined by the cut:

\[ K = \{ q \in \mathbb{Q} \mid q^2 < 2 \} \cup \{ q \in \mathbb{Q} \mid q < 0 \}. \]

(Note: One can also define multiplication of cuts, but its a lot more tedious because of negative numbers. Try it for yourself if you are interested.)

**Problem # 3**

In each of the following cases find the limit of the sequence and give a formal proof of convergence. That is give explicitly \( N(\epsilon) \) for each \( 0 < \epsilon \leq 1 \) in the problems listed below. The bound you prove does not have to be the best possible, but your choices of \( N(\epsilon) \) do need to work for each \( 0 < \epsilon \leq 1 \).

a) \( \lim \frac{n \sin(n) + 1}{n^2 + 1} \).

b) \( \lim \frac{1 + a_n^2}{b_n + 1} \) where \( |a_n - 1| \leq \frac{1}{n} \) and \( |b_n - 1| \leq \frac{1}{n} \).

c) \( \lim \frac{n}{n^n} \).