MATH 142A HOMEWORK 2

Important: Please answer each of these questions on a separate sheet(s) of paper. Also, put your name and section number on each sheet. You will then upload your final solutions GradeScope as explained on the class webpage.

Problem # 1

In each of the following cases compute the limit of the sequence and give a formal proof of convergence, i.e. give explicitly $N(\epsilon)$ for each $0 < \epsilon \leq 1$. You do not have to be precise, but your choices of $N(\epsilon)$ do need to work for each $0 < \epsilon \leq 1$.

a) $\lim \frac{\sin(n) + \cos(2n)}{\ln(1 + n)}$.

b) $\lim \frac{1 + a_n^2}{b_n}$ where $|a_n - 1| \leq \frac{1}{n}$ and $|b_n - 10| \leq \frac{1}{n}$.

c) $\lim \frac{n^2}{n^n}$.

Problem # 2

These questions provide some useful rules for dealing with limits.

a) Prove that if $a_n$ converges with $\lim a_n = L$, and $b_n$ is any other sequence with $\lim(a_n - b_n) = 0$, then $b_n$ converges and $\lim b_n = L$ as well.

b) Let $a_n$, $c_n$ be convergent sequences with $L = \lim a_n = \lim c_n$. Prove that if $b_n$ is any other sequence with $a_n \leq b_n \leq c_n$ for all $n$, then $\lim b_n = L$.

c) Let $a_n$, $b_n$, be converging sequences with the property that $a_n \leq b_n$ for infinitely many (but possibly not all) $n$. Prove that $\lim a_n \leq \lim b_n$.

Problem # 3

These problems concern approximating arbitrary real numbers by certain kinds of sequences.

a) Prove that for every real number $r \in \mathbb{R}$ there exists a sequence of rational numbers $q_1 < q_2 < \ldots < r$ with the property $q_n \to r$.

b) Prove that for every real number $r \in \mathbb{R}$ there exists a sequence of irrational numbers $z_1 < z_2 < \ldots < r$ with the property $z_n \to r$.

Problem #4

Let $a_n > 0$ be a sequence of strictly positive numbers. Suppose that $\alpha = \lim \frac{a_{n+1}}{a_n}$ exists.

a) Prove that if $\alpha < 1$ then $a_n \to 0$.

b) Prove that if $\alpha > 1$ then $a_n \to \infty$.

c) Give an example with $\alpha = 1$ and $a_n \to 0$.

d) Give an example with $\alpha = 1$ and $a_n \to \infty$.

e) Give an example with $\alpha = 1$ and $a_n$ bounded but not converging to any limit at all.
Problem #5

Let \( a_1, a_2, \ldots a_N \) be a collection of real numbers. A \textit{weighted average} is an expression of the form:

\[
A_N = \frac{k_1a_1 + k_2a_2 + \ldots + k_Na_N}{k_1 + k_2 + \ldots + k_N},
\]

where all \( k_i \in \mathbb{N} \). Note that the case \( k_1 = k_2 = \ldots = k_N = 1 \) corresponds to the usual notion of average.

a) Prove that if \( a_n \to L \) then the weighted averages above are such that \( A_N \to L \) regardless of how the weights \( k_i \) are chosen.

b) Show that a similar result is true when \( a_n \to +\infty \).