MATH 142A HOMEWORK 3

**Important:** Please answer each of these questions on a separate sheet(s) of paper. Also, put your name and section number on each sheet. You will then upload your final solutions GradeScope as explained on the class webpage.

**Problem #1**

These questions provide some useful rules for dealing with limits.

a) Prove that if \( a_n \) converges with \( \lim a_n = L \), and \( b_n \) is any other sequence with \( \lim(a_n - b_n) = 0 \), then \( b_n \) converges and \( \lim b_n = L \) as well.

b) Let \( a_n, c_n \) be convergent sequences with \( L = \lim a_n = \lim c_n \). Prove that if \( b_n \) is any other sequence with \( a_n \leq b_n \leq c_n \) for all \( n \), then \( \lim b_n = L \).

c) Let \( a_n, b_n \), be converging sequences with the property that \( a_n \leq b_n \) for infinitely many (but possibly not all) \( n \). Prove that \( \lim a_n \leq \lim b_n \).

**Problem #2**

Let \( a_n > 0 \) be a sequence of strictly positive numbers. Suppose that \( \alpha = \lim \frac{a_{n+1}}{a_n} \) exists.

a) Prove that if \( \alpha < 1 \) then \( a_n \to 0 \).

b) Prove that if \( \alpha > 1 \) then \( a_n \to \infty \).

c) Give an example with \( \alpha = 1 \) and \( a_n \to 0 \).

d) Give an example with \( \alpha = 1 \) and \( a_n \to \infty \).

e) Give an example with \( \alpha = 1 \) and \( a_n \) bounded but not converging to any limit at all.

**Problem #3**

Let \( a_1, a_2, \ldots, a_N \) be a collection of real numbers. A *weighted average* is an expression of the form:

\[
A_N = \frac{k_1a_1 + k_2a_2 + \cdots + k_N a_N}{k_1 + k_2 + \cdots + k_N},
\]

where all \( k_i \in \mathbb{N} \). Note that the case \( k_1 = k_2 = \ldots = k_N = 1 \) corresponds to the usual notion of average.

a) Prove that if \( a_n \to L \) then the weighted averages above are such that \( A_N \to L \) regardless of how the weights \( k_i \) are chosen.

b) Show that a similar result is true when \( a_n \to +\infty \).

**Problem #4**

Start with two real numbers \( 0 \leq a \leq b \) and define sequences \( a_n \) and \( b_n \) by:

\[
a_1 = a, \quad b_1 = b, \quad \text{and} \quad a_n = \sqrt{a_{n-1}b_{n-1}}, \quad b_n = \frac{a_{n-1} + b_{n-1}}{2} \quad \text{when} \quad n \geq 2.
\]

Prove that \( a_n \) both \( b_n \) converge and furthermore prove that \( \lim a_n = \lim b_n \).