Important: Please answer each of these questions on a separate sheet(s) of paper. You will then upload your final solutions GradeScope as explained on the class webpage.

Problem #1

Let \( f : [a, b] \to [c, d] \) be a continuous function. Suppose that \( g : [a, b] \to \mathbb{R} \) is another continuous function such that there exists \( x_1, x_2 \in [a, b] \) with the property \( g(x_1) = c \) and \( g(x_2) = d \). Prove that there exists some \( x_0 \in [a, b] \) with \( f(x_0) = g(x_0) \).

Problem #2

Let \( f : [a, b] \to [a, b] \) be a continuous function with \( |f(x) - f(y)| < |x - y| \) for all \( x \neq y \in [a, b] \). Prove that there exists a unique \( x_0 \in [a, b] \) with \( f(x_0) = x_0 \), and moreover that the iterated composition \( f^n = f \circ f \circ \ldots \circ f \) (n times) is such that \( x_n = f^n(x) \to x_0 \) for all \( x \in [a, b] \).

Problem #3

Consider the following function \( f : \mathbb{R} \to \mathbb{R} \):

\[
 f(x) = \begin{cases} 
 1, & \text{if } x = 0; \\
 \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ where } m \in \mathbb{Z}, n \in \mathbb{N} \text{ and } \gcd(n, m) = 1; \\
 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}.
\end{cases}
\]

Show that \( f \) is continuous at all \( x \in \mathbb{R} \setminus \mathbb{Q} \), and discontinuous at all \( x \in \mathbb{Q} \).

Problem #4

Let \( p(x) : \mathbb{R} \to \mathbb{R} \) be a monic odd degree polynomial, i.e. \( p(x) = \sum_{k=0}^{d} a_k x^k \) where \( d \) is odd and \( a_d = 1 \). Suppose there exists \( x_1 < x_2 \) with \( p(x_2) < 0 < p(x_1) \). Show that \( p(x) \) has at least three distinct real roots.

Show that the polynomial \( p(x) = x^3 + 4x^2 + x - 1 \) factors with all real roots.

Problem #5

Let \( f : D \to \mathbb{R} \), and let \( x_0 \in D \). We say “\( f \) is left continuous at \( x_0 \)” if whenever \( x_n \in D \) and \( x_n \leq x_0 \), then \( x_n \to x_0 \) implies \( f(x_n) \to f(x_0) \). Likewise we say “\( f \) is right continuous at \( x_0 \)” if whenever \( x_n \in D \) and \( x_n \geq x_0 \), then \( x_n \to x_0 \) implies \( f(x_n) \to f(x_0) \).

Show that a function \( f : D \to \mathbb{R} \) is continuous at \( x_0 \in D \) iff it is both right and left continuous as \( x_0 \).

Problem #6

A function \( f : D \to \mathbb{R} \) is called lower semicontinuous at \( x_0 \in D \) if \( f(x_0) \leq \liminf f(x_n) \) for all sequences \( x_n \in D \) with \( x_n \to x_0 \in D \).

A function \( f : D \to \mathbb{R} \) is called upper semicontinuous at \( x_0 \in D \) if \( \limsup f(x_n) \leq f(x_0) \) for all sequences \( x_n \in D \) with \( x_n \to x_0 \in D \).

a) Prove that a function \( f : D \to \mathbb{R} \) is continuous at \( x_0 \in D \) iff it is both upper and lower semicontinuous at \( x_0 \).

b) Prove that a monotone increasing left continuous function (see Problem #5 above) is always lower semicontinuous.

c) Prove that a monotone increasing right continuous function (see Problem #5 above) is always upper semicontinuous.

d) Prove that the function from Problem #3 above is upper semicontinuous at all points.