MATH 142A HOMEWORK 7

**Important:** Please answer each of these questions on a separate sheet(s) of paper. You will then upload your final solutions GradeScope as explained on the class webpage.

**Problem # 1**

Let \( f : [a, b] \to [c, d] \) be a continuous function. Suppose that \( g : [a, b] \to \mathbb{R} \) is another continuous function such that there exists \( x_1, x_2 \in [a, b] \) with the property \( g(x_1) = c \) and \( g(x_2) = d \). Prove that there exists some \( x_0 \in [a, b] \) with \( f(x_0) = g(x_0) \).

**Problem # 2**

Let \( f : [a, b] \to [a, b] \) be a continuous function with \( |f(x) - f(y)| < \gamma |x - y| \) for all \( x \neq y \in [a, b] \) and some \( 0 \leq \gamma < 1 \). Prove that there exists a unique \( x_0 \in [a, b] \) with \( f(x_0) = x_0 \), and moreover that the iterated composition \( f^n = f \circ f \circ \ldots \circ f \) (\( n \) times) is such that \( x_n = f^n(x) \to x_0 \) for all \( x \in [a, b] \).

**Problem # 3**

Consider the following function \( f : \mathbb{R} \to \mathbb{R} \):

\[
 f(x) = \begin{cases} 
 1, & \text{if } x = 0; \\
 \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ where } m \in \mathbb{Z}, n \in \mathbb{N} \text{ and } \gcd(n, m) = 1; \\
 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. 
\end{cases}
\]

Show that \( f \) is continuous at all \( x \in \mathbb{R} \setminus \mathbb{Q} \), and discontinuous at all \( x \in \mathbb{Q} \).

**Problem # 4**

Let \( p(x) : \mathbb{R} \to \mathbb{R} \) be a monic odd degree polynomial, i.e. \( p(x) = \sum_{k=0}^{d} a_k x^k \) where \( d \) is odd and \( a_d = 1 \).

a) Suppose there exists \( x_1 < x_2 \) with \( p(x_2) < 0 < p(x_1) \). Show that \( p(x) \) has at least three distinct real roots.

b) Show that the polynomial \( p(x) = x^3 + 4x^2 + x - 1 \) factors with all real roots.