MATH 142A MIDTERM 1

**Important:** Please put your name on each sheet, and your student ID number on the top sheet. Please show all your work on the pages provided. These will be uploaded to GradeScope. You may use both sides of each sheet. Use the last sheet if you need more room.

**Part I (30 pts. total)**

For each of the following give a complete definition:

a) (10 pts.) Let \( S \subseteq \mathbb{R} \) be a nonempty set of real numbers. Define what it means for a number \( L \in \mathbb{R} \) to be the “least upper bound” for \( S \). In other words give a precise definition of the relation \( L = \text{SUP}(S) \).

**Solution:** The least upper bound \( L = \text{SUP}(S) \) is defined to be the number which satisfies the following two properties:

(a) \( L \) is an upper bound for \( S \). That is \( x \leq L \) for all \( x \in S \).

(b) \( L \) is the smallest upper bound for \( S \). That is \( x \leq M \) for all \( x \in S \) implies \( L \leq M \).

b) (10 pts.) Give a precise statement of the “completeness axiom” of the real number system.

**Solution:** The “completeness axiom” of the real number system states: “Let \( S \subseteq \mathbb{R} \) be any nonempty subset of real numbers which is bounded above. Then \( S \) has a least upper bound which is a real number.”

c) (10 pts.) Let \( a_n \in \mathbb{R} \) be a sequence of real numbers. Define what it means for \( a_n \) to “converge to \(+\infty\)”. In other words give the precise definition of \( a_n \rightarrow +\infty \).

**Solution:** We say \( a_n \rightarrow +\infty \) is given any \( M > 0 \) there exists \( N \in \mathbb{N} \), such that for all \( n > N \) one has \( a_n > M \).
Part II (40 pts. total)

For each of the following give a short answer. Please be careful to state clearly any results from lectures or the book you use. If the answer is TRUE give a short explanation (e.g. a short calculation plus a theorem or definition from the book or class). If the answer is FALSE give an explicit counterexample.

a) (10 pts.) TRUE or FALSE: If $0 < a, b, c, d$ are positive real numbers with $a \leq b$ and $c \leq d$ then one has $\frac{a}{d} \leq \frac{b}{c}$.

**Solution:** TRUE. If $a \leq b$ and $\lambda \geq 0$ then $\lambda a \leq \lambda b$. Applying this and the conditions $0 < a \leq b$ and $0 < c \leq d$ we have the chain of inequalities:

$$ac \leq bc \leq bd.$$ 

Multiplying both sides of $ac \leq bd$ by the non-negative quantity $\frac{1}{cd}$ we get $\frac{a}{d} \leq \frac{b}{c}$ as desired.

b) (10 pts.) TRUE or FALSE: For every real $\epsilon > 0$ there exists a natural number $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

**Solution:** TRUE. This follows immediately from the Archimedean Property of real numbers which states that for each $r \in \mathbb{R}$ there exists $n \in \mathbb{N}$ with $r < n$. Applying this to $r = \frac{1}{\epsilon}$ we get $\frac{1}{\epsilon} < n$. Flipping the inequality gives $\frac{1}{n} < \epsilon$. 


c) (10 pts.) TRUE or FALSE: If \( a_n \neq 0 \) is a sequence of real numbers and \( a_n \rightarrow 0 \) then \( \frac{1}{|a_n|} \rightarrow +\infty \).

**Solution:** TRUE. This follows from a limit theorem proved in class, namely that if \( b_n > 0 \) then \( b_n \rightarrow \infty \) iff \( \frac{1}{b_n} \rightarrow 0 \). Applying this to \( b_n = |a_n| \) we just need to show \( a_n \rightarrow 0 \) implies \( |a_n| \rightarrow 0 \). But this follows directly from the definition of convergence to 0 because \( ||a_n| - 0| = |a_n - 0| \). Thus \( |a_n - 0| < \epsilon \) for all \( n > N \) immediately implies the same thing for \( |a_n| \) as well.

d) (10 pts.) Let \( a_n \) and \( b_n \) be sequences of real numbers with \( a_n \rightarrow 10 \) and \( b_n \rightarrow 1 \). Compute:

\[
\lim_{n \rightarrow \infty} \frac{2(a_n + n)^2}{(nb_n)^2 + 1}.
\]

You do not have to use the \( \epsilon, N \) definition here, but please be sure to state carefully any limits theorems from class you are using.

**Solution:** First divide by \( n^2 \) in the numerator and denominator to write the limit as:

\[
\lim_{n \rightarrow \infty} \frac{2 + 2a_n/n + a_n^2/n^2}{b_n^2 + 1/n^2}.
\]

Now let \( c_n = a_n/n = a_n \cdot \frac{1}{n} \). Then we have \( c_n \rightarrow 0 \) because of the product rule for limits and \( a_n \rightarrow 10 \) and \( \frac{1}{n} \rightarrow 0 \). Then if \( p(x) = x^2 + 2x + 1 \) by again applying product and sum rules we have \( p(c_n) \rightarrow p(0) = 2 \).

This shows that the limit of the numerator above is 2.

Likewise for the denominator we have \( b_n^2 + \frac{1}{n^2} \rightarrow 1 \) by sum and product rules.

Finally, by the ratio rule for limits we have:

\[
\lim_{n \rightarrow \infty} \frac{2 + 2a_n/n + a_n^2/n^2}{b_n^2 + 1/n^2} = \frac{\lim_{n \rightarrow \infty} (2 + 2a_n/n + a_n^2/n^2)}{\lim_{n \rightarrow \infty} (b_n^2 + 1/n^2)} = \frac{2}{1} = 2.
\]

**Note:** In problem d) above directly exchanging \( a_n \) for 10 and \( b_n \) for 1 and then taking the limit as \( n \rightarrow \infty \) does lead to the correct answer, but as the next example shows such manipulations are not valid in general. Consider:

\[
\lim_{n \rightarrow \infty} \frac{(a_n + n)^2 - n^2}{b_n^2 + 1}, \quad \text{where} \quad a_n \rightarrow 0, \quad b_n \rightarrow 1.
\]

Sticking in the limits for \( a_n \) and \( b_n \) directly would give:

\[
\lim_{n \rightarrow \infty} \frac{(0 + n)^2 - n^2}{1^2 + 1} = \lim_{n \rightarrow \infty} 0 = 0.
\]

On the other hand if \( a_n = \frac{1}{n} \) we see that the original limit is actually computed to be:

\[
\lim_{n \rightarrow \infty} \frac{(a_n + n)^2 - n^2}{b_n^2 + 1} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n} + n\right)^2 - n^2}{b_n^2 + 1} = \lim_{n \rightarrow \infty} \frac{2 + \frac{1}{n^2}}{b_n^2 + 1} = \frac{2}{1^2 + 1} = 1.
\]

This shows that handling infinite limits can be tricky in general. The safe thing to do is manipulate the problem so that standard (finite) limit theorems can be applied.
PART III (30 pts. total)

For each of the following give a complete proof:

a) (15 pts) Let \( S \subseteq \mathbb{R} \) be a bounded and nonempty subset of real numbers. Show that:
\[
SUP(S) = -INF(-S).
\]
Here \(-S = \{ -x \mid x \in S \}\) is the set of all numbers in \( S \) multiplied by \(-1\).

**Solution:** First, by definition of \( SUP(S) \) we have \( x \leq SUP(S) \) for all \( x \in S \). Multiplying this inequality by \(-1\) yields \( y \geq -SUP(S) \) for all \( y \in -S \). In other words \(-SUP(S)\) is a lower bound for \(-S\), so by definition of the greatest lower bound for \(-S\) we must have:
\[
-SUP(S) \leq INF(-S).
\]
On the other hand for all \( y \in -S \) we have \( INF(-S) \leq y \), which yields \( x \leq -INF(-S) \) for all \( x \in S \). Thus \(-INF(-S)\) is an upper bound for \( S \), so by definition of least upper bound for \( S \) we must have:
\[
SUP(S) \leq -INF(-S).
\]
We have shown that both \(-INF(-S) \leq SUP(S)\) and \( SUP(S) \leq -INF(-S)\), so the identity \( SUP(S) = -INF(-S) \) is established.

b) (15 pts) Use mathematical induction to show that for any real \( a \neq 1 \) one has identity:
\[
(1 + a + a^2 + \ldots + a^n) = \frac{1 - a^{n+1}}{1 - a}.
\]
(Note: You need to use induction to receive credit for this part.) Next, use the above identity and the limit theorems we proved in class to show that for \(|a| < 1\) one has:
\[
\lim_{n \to \infty} (1 + a + a^2 + \ldots + a^n) = \frac{1}{1 - a}.
\]

**Solution:** For the inductive part note that when \( a \neq 1 \) we have \( 1 + a = \frac{(1+a)(1-a)}{1-a} = \frac{1-a^2}{1-a} \), which directly establishes the identity when \( n = 1 \). Next, if we assume by induction that \((1 + a + \ldots + a^n) = \frac{1 - a^{n+1}}{1 - a}\), for the \( n + 1 \) stage we calculate:
\[
(1+a+\ldots+a^n+a^{n+1}) = \frac{1 - a^{n+1}}{1 - a} + a^{n+1} = \frac{1 - a^{n+1} + a^{n+1}(1 - a)}{1 - a} = \frac{1 - a^{n+1} + a^{n+1} - a^{n+2}}{1 - a} = \frac{1 - a(n+1) + a^{n+1}}{1 - a}.
\]
To show the limit part we use that if \(|a| < 1\) then \( \lim_{n \to \infty} a^{n+1} = 0 \) (note that shifting the index does not change the limit). This combined with the limit theorem for ratios and sums gives:
\[
\lim_{n \to \infty} (1 + a + a^2 + \ldots + a^n) = \lim_{n \to \infty} \frac{1 - a^{n+1}}{1 - a} = \lim_{n \to \infty} (1 - a^{n+1}) = \frac{1 - \lim_{n \to \infty} a^{n+1}}{1 - a} = \frac{1}{1 - a}.
\]
Use this page to show your work. Please label any additional work here by the corresponding problem number.