MATH 142A MIDTERM 2

Important: Please put your name on each sheet, and your student ID number on the top sheet. Please show all your work on the pages provided. These will be uploaded to GradeScope. You may use both sides of each sheet. Use the last sheet if you need more room.

Part I (30 pts. total)

For each of the following give a complete definition:

a) (10 pts.) Let $f : D \rightarrow \mathbb{R}$, and fix $x_0 \in D$. Give the sequential definition of “$f$ is continuous at $x_0$”.

Solution: $f : D \rightarrow \mathbb{R}$ is said to be continuous at $x_0 \in D$ if for every sequence $x_n \in D$ with $x_n \rightarrow x_0$ one has $f(x_n) \rightarrow f(x_0)$.

b) (10 pts.) Let $(a_k)$ be a sequence of real numbers defined for $k \geq k_0$. Define what it means for the series $\sum_{k=k_0}^{\infty} a_k$ to satisfy the “Cauchy criterion”. (For this question you must give a complete definition involving $\epsilon$.)

Solution: We say that the series $\sum_{k=k_0}^{\infty} a_k$ satisfies the Cauchy criterion if for every $\epsilon > 0$ there exists an $N = N(\epsilon) \in \mathbb{N}$ with $|\sum_{m=m}^{n} a_k| < \epsilon$ whenever $n \geq m > N$ (WLOG assume $N \geq k_0$).

c) (10 pts.) Let $(a_n)$ be a sequence of real numbers. Define what it means for a real number $a$ to be a “subsequential limit of $(a_n)$”.

Solution: We say that $a \in \mathbb{R}$ is a subsequential limit of sequence $(a_n)$ if there exists as subsequence $(a_{n_k})$ with the property that $a_{n_k} \rightarrow a$. 

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PART II (40 pts. total)

For each of the following give a short answer. Please be careful to state clearly any results from lectures or the book you use. If the answer is TRUE give a short explanation (e.g. a short calculation plus a theorem or definition from the book or class). If the answer is FALSE give an explicit counterexample.

a) (10 pts.) TRUE or FALSE: There exists a sequence \((a_n)\) of real numbers such that its set of subsequential limits is exactly:

\[
L(a_n) = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\}.
\]

**Solution:** FALSE. A theorem of the text says that if \(L(a_n)\) is the set of subsequential limits of \((a_n)\), and if \(b_k \in L(a_n)\) is another sequence with \(b_k \to b\), then one must also have \(b \in L(a_n)\) as well (in other words \(L(a_n)\) is closed under taking limits). Now \(b_k = 1/k\) is a sequence in \(\left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}\), and \(b_k \to 0\). But 0 is not in the set \(\left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}\), so this set cannot possibly be the set of subsequential limits of a sequence \((a_n)\).

b) (10 pts.) TRUE or FALSE: Suppose that \(a_k \geq 0\), and \(\sum_{k=1}^{\infty} a_k\) converges. Then \(\sum_{k=1}^{n} (-1)^k a_k\) converges.

**Solution:** TRUE. This follows immediately from the comparison test and the fact \(|(-1)^k a_k| \leq a_k\).
c) (10 pts.) TRUE or FALSE: The series \( \sum_{k=1}^{\infty} 10^k 2^{-k^2} \) converges.

**Solution:** TRUE. Let \( a_k = 10^k 2^{-k^2} \) denote the terms of this series. Then \( \sqrt[a_k]{a_k} = 10 \cdot 2^{-k} \) and \( (1/2)^k \to 0 \) from another theorem of the text. Thus \( \lim_k \sqrt[a_k]{a_k} = 0 < 1 \) so the series converges by the root test.

d) (10 pts.) TRUE or FALSE: If \( f : [a, b] \to \mathbb{R} \) is continuous and \( f(x) > 0 \) for all \( x \in [a, b] \), then the function \( 1/f(x) : [a, b] \to \mathbb{R} \) is bounded.

**Solution:** TRUE. By a theorem of the text if \( f : D \to \mathbb{R} \) is continuous, and \( f(x) \neq 0 \) for all \( x \in D \), then \( 1/f(x) : D \to \mathbb{R} \) is also continuous. Thus \( 1/f : [a, b] \to \mathbb{R} \) is continuous in this case. By another theorem from the text any continuous function on a closed and bounded interval \( [a, b] \subset \mathbb{R} \) must be bounded.
PART III (30 pts. total)

For each of the following give a complete proof:

a) (15 pts) Let \(a_n\) be defined recursively by the formula:
\[
a_1 = 2, \quad a_n = \frac{1}{2}(1 + a_{n-1}) \text{ when } n > 1.
\]
Show that \(a_n\) converges and find \(\lim_{n \to \infty} a_n\).

**Solution:** First we’ll show that \(a_n\) is monotone decreasing by induction: We have \(a_1 = 2\) and \(a_2 = \frac{3}{2}\) so clearly \(a_2 < a_1\). Assuming \(a_{n-1} \leq a_n\) we compute \(a_{n+1} = \frac{1}{2}(1 + a_n) \leq \frac{1}{2}(1 + a_{n-1}) = a_n\).

In addition to this another simple induction shows \(0 \leq a_n\) for all \(n\). Thus, by the monotone convergence theorem we know that \(a_n \to a\) for some \(a \in \mathbb{R}\).

Finally let’s compute the limit \(a\). By the identity \(a_n = \frac{1}{2}(1 + a_{n-1})\) and basic limit rules we must have
\[
a = \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{2}(1 + a_{n-1}) = \frac{1}{2}(1 + \lim_{n \to \infty} a_{n-1}) = \frac{1}{2}(1 + a).
\]
Solving for \(a\) in this equation gives \(a = 1\).

**Alternate Solution:** It is possible to solve for \(a_n\) in closed form. We have:
\[
a_n - 1 = \frac{1}{2}(1 + a_{n-1}) - 1 = \frac{1}{2}(a_{n-1} - 1),
\]
and also \(a_1 - 1 = 1\). So assuming \(a_n - 1 = 2^{1-n}\) gives
\[
a_{n+1} - 1 = \frac{1}{2}(a_n - 1) = 2^{-n}.
\]
Thus, the formula \(a_n = 1 + 2^{1-n}\) is established by induction. The conclusion \(a_n \to 1\) follows immediately from limit theorems of the text.

b) (15 pts) Show that there exists a number \(0 < x < 1\) which satisfies the equation \(x = (1 - x^2)^{2018}\).

**Solution:** This is a standard application of the intermediate value theorem. Define the function
\[
f(x) = x - (1 - x^2)^{2018}.
\]
Then \(f : [0, 1] \to \mathbb{R}\) is continuous (its a polynomial), and \(f(0) = -1\) while \(f(1) = 1\). Since \(-1 < 0 < 1\) there must exist some \(x_0 \in [0, 1]\) with \(f(x_0) = 0\). Clearly \(x_0 \neq 0, 1\) so we must have \(0 < x_0 < 1\) for this root of \(f(x)\).
Use this page to show your work. Please label any additional work here by the corresponding problem number.