MATH 142B HOMEWORK 3

Important: Please answer each of these questions on a separate sheet(s) of paper. Also, put your name and section number on each sheet. You will then upload your final solutions GradeScope as explained on the class webpage.

Problem # 1

Consider the function \( f(x) = \frac{1}{1+x^2} \) which is defined for all \( x \in \mathbb{R} \).

a) Find the radius of convergence for the Taylor series of \( f(x) \) centered at the point \( x_0 = 0 \).

b) Show that \( f(x) \) is an analytic function on all of \( \mathbb{R} \).

Problem # 2

Let \( a_k, b_k \in \mathbb{R} \) be sequences. A function series of the form:

\[
\sum_{k=0}^{\infty} \left( a_k \sin(kx) + b_k \cos(kx) \right)
\]

is called a “Fourier series”.

a) Show that if one assumes \( \sum_{k=0}^{\infty} |a_k| < \infty \) and \( \sum_{k=0}^{\infty} |b_k| < \infty \) then the Fourier series above converges uniformly on all of \( \mathbb{R} \) to a continuous function \( f(x) \) with the property that \( f(x+2\pi) = f(x) \) for all \( x \in \mathbb{R} \).

b) Show that if one assumes the stronger condition \( \sum_{k=1}^{\infty} k|a_k| < \infty \) and \( \sum_{k=1}^{\infty} k|b_k| < \infty \), then the Fourier series above converges to a differentiable function.

Problem # 3

Let \( (a_k)_{k=0}^{\infty} \) be a sequence of real numbers, and denote their partial sums by \( S_n = \sum_{k=0}^{n} a_k \). Assume that the sequence of averages \( A_n = \frac{S_n}{n+1} = \frac{a_0 + a_1 + \ldots + a_n}{n+1} \) is uniformly bounded.

a) Show that both the power series \( \sum_{k=0}^{\infty} a_k x^k \) and \( \sum_{n=0}^{\infty} S_n x^n \) converge for \( |x| < 1 \) and that one has the identity:

\[
\sum_{k=0}^{\infty} a_k x^k = (1-x) \sum_{n=0}^{\infty} S_n x^n .
\]

b) Suppose in addition that the averages converge \( \frac{S_n}{n+1} = \frac{a_0 + a_1 + \ldots + a_n}{n+1} \to A \). Then one has the limit:

\[
\lim_{x \to 1^-} (1-x) \sum_{k=0}^{\infty} a_k x^k = A .
\]

c) Prove that \( \sum_{n=0}^{\infty} 2^n x^n \) is a convergent power series with radius of convergence \( R = 1 \), and that there exists a \( C > 0 \) such that:

\[
|f(x)| \leq \frac{C}{1-|x|}, \quad \text{when} \quad |x| < 1 .
\]