MATH 142B HOMEWORK

**Important:** Please answer each of these questions on a separate sheet(s) of paper. Also, put your name and section number on each sheet. You will then upload your final solutions GradeScope as explained on the class webpage.

**Problem # 1**

Let \( f_n : [a, b] \to \mathbb{R} \) be a sequence of Riemann integrable functions, and suppose that \( f_n \to f \) uniformly on \([a, b]\). Prove that \( f \) is Riemann integrable and that one has:

\[
\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx .
\]

**Problem # 2**

Let \( f : [a, b] \to \mathbb{R} \) be Riemann integrable.

a) Show that \( f^2 \) is Riemann integrable.

b) Show by induction that \( p(f) \) is Riemann integrable where \( p(y) = \sum_{k=0}^{n} a_k y^k \) is any polynomial.

c) Let \( f([a, b]) \subseteq [c, d] \) and suppose that \( G : [c, d] \to \mathbb{R} \) is any continuous function. Show that the composition \( G(f) : [a, b] \to \mathbb{R} \) is Riemann integrable. (Hint: There are several ways to do this. One is to use previous work and the Weierstrass approximation theorem.)

**Problem # 3**

Let \( f : [a, b] \to \mathbb{R} \) be Riemann integrable, and let \( x_n \in [a, b] \) be any set of finitely many distinct points (i.e. \( x_n \neq x_m \) for \( n \neq m \), and \( n = 1, \ldots, N \)), and \( y_n \in \mathbb{R} \) any other finite collection of points for the same indices \( n = 1, \ldots N \). Define:

\[
g(x) = \begin{cases} f(x), & x \neq x_n \text{ for all } n; \\ y_n, & x = x_n \text{ for some } n. \end{cases}
\]

Prove that \( g \) is also Riemann integrable, and that one has the formula:

\[
\int_a^b f(x) \, dx = \int_a^b g(x) \, dx .
\]

In other words changing a Riemann integrable function on a finite set does not change the integral.

What can happen when one changes \( f \) on an infinite but countable set?

**Problem # 4**

A function \( f : (a, b) \to \mathbb{R} \) is said to have a “jump discontinuity” at \( x_0 \in (a, b) \) if both \( \lim_{x \to x_0^-} f(x) = L^- \) and \( \lim_{x \to x_0^+} f(x) = L^+ \) exist but \( L^- \neq L^+ \). Suppose that \( f : [a, b] \to \mathbb{R} \) is Riemann integrable and has a jump discontinuity at some \( x_0 \in (a, b) \). Prove that \( F(x) = \int_a^x f(t) \, dt \) cannot be differentiable at \( x = x_0 \).
PROBLEM # 5

Give an example of an integrable function $f : [a, b] \to \mathbb{R}$ which has a discontinuity at some $x_0 \in (a, b)$ and is such that $F(x) = \int_a^x f(t)dt$ is differentiable at every point $x \in (a, b)$. 