Important: Please put your name and your student ID number on the top of the sheet. Please show all your work. You may use both sides of the sheet.

**Question I (5 pts)**

Let $f_n : S \to \mathbb{R}$ be a sequence of functions, and let $f : S \to \mathbb{R}$ be another function. Give a precise definition for $f_n \to f$ uniformly on $S$.

Let $\epsilon > 0$ be given. Then there exists an $N = N(\epsilon) \in \mathbb{N}$ such that for all $n > N$ one has:

$$|f_n(x) - f(x)| < \epsilon, \quad \text{for all} \quad x \in S.$$

**Question II (5 pts total)**

Let $f_n : [a, b] \to \mathbb{R}$, and assume that $f_n \to f$ uniformly on $[a, b]$.

a) (4 pts) Assume that each $f_n : [a, b] \to \mathbb{R}$ is continuous. Show that if $x_n \in [a, b]$ is any sequence with $x_n \to x$, then $f_n(x_n) \to f(x)$.

Let $\epsilon > 0$. Choose $N_1$ so that whenever $n > N_1$ we get:

$$|f_n(y) - f(y)| < \frac{\epsilon}{2}, \quad \text{for all} \quad y \in [a, b].$$

Next, choose $N_2$ so that whenever $n > N_2$ we get:

$$|f(x_n) - f(x)| < \frac{\epsilon}{2}.$$

Now let $N = \max\{N_1, N_2\}$. Then for all $n > N$ we have:

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

b) (1 pt) Give an example to show that if $f_n \to f$ only pointwise, then one can have a sequence $x_n \to x$ but $f_n(x_n)$ does not necessarily converge to $f(x)$.

Let $S = [0, 1]$ and:

$$f_n(x) = x^n, \quad f(x) = \begin{cases} 0, & 0 \leq x < 1; \\ 1, & x = 1. \end{cases}$$

Then $f_n \to f$ pointwise. In addition if $x_n = 2^{-\frac{n}{2}}$ we have $x_n \to 1$. But $f_n(x_n) = \frac{1}{2}$ for all $n$, so we have $f_n(x_n) \to \frac{1}{2} \neq 1 = f(1)$. 