INTRODUCTION TO ANALYSIS ON $\mathbb{R}^n$

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1. The Euclidean Distance

We denote points in the vector space $\mathbb{R}^d$ by $x = (x_1, \ldots, x_d)$ where $d \in \mathbb{N}$ is the dimension. Recall that if $a, b, c$ are the side lengths of a right triangle in $\mathbb{R}^2$, then $a^2 + b^2 = c^2$ where $c$ is the hypotenuse (i.e. the one opposite to the right angle). The vector analog of this identity is that if one defined the quantity:

$$
\| x \| = \left(\sum_{i=1}^{d} x_i^2\right)^{\frac{1}{2}},
$$

then $\| x \|$ measures the Euclidean distance between $x$ and the origin $\vec{0} = (0, \ldots, 0)$. Rigidly translating this to other vectors $y$ means that $\| x - y \|$ is the distance from $x$ to $y$ along the ray $x - y$.

If $T_1, T_2, T_3$ denote the three side lengths of any triangle, then one always has $T_3 \leq T_1 + T_2$. In particular if the triangle has vertices $\vec{0}, x, y$ then one should expect $\| x - y \| \leq \| x \| + \| y \|$. To see this is indeed the case for the Euclidean distance as we have defined it we first prove:

**Proposition 1.1** (Cauchy-Schwarz Inequality). For two vectors $x, y \in \mathbb{R}^d$ define their dot product as the quantity $x \cdot y = \sum_{i=1}^{d} x_i y_i$. Then one has:

$$
|x \cdot y| \leq \| x \| \| y \|.
$$

**Proof.** If one of $\| x \|$ or $\| y \|$ is zero this is trivial. Otherwise by dividing through by $\| x \|$ and $\| y \|$ we may assume both $\| x \| = \| y \| = 1$, in which case we are trying to show $|x \cdot y| \leq 1$. Using $|x_i y_i| \leq \frac{1}{2}(x_i^2 + y_i^2)$ we indeed are able to compute that:

$$
|x \cdot y| \leq \sum_{i=1}^{d} |x_i y_i| \leq \frac{1}{2} \sum_{i=1}^{d} (x_i^2 + y_i^2) = 1.
$$

This immediately leads to:

**Proposition 1.2** (Triangle Inequality). Let $x, y \in \mathbb{R}^d$ be any two vectors. Then $\| x + y \| \leq \| x \| + \| y \|$. 

**Proof.** Writing things we see:

$$
\| x + y \|^2 = \| x \|^2 + 2 x \cdot y + \| y \|^2 \leq \| x \|^2 + 2 \| x \| \| y \| + \| y \|^2 = (\| x \| + \| y \|)^2.
$$

Taking the square root of both sides we are done. 

1.1. Convergence in $\mathbb{R}^d$. Using the Euclidean distance function, we have a notion of sequences of vectors converging in $\mathbb{R}^d$. Because of the subindex notation for coordinates, we’ll use a superindex notation for elements of a sequence and write $x^{(n)} = (x^{(n)}_1, \ldots, x^{(n)}_d)$. In other words one can think if a sequence of vectors as a $d$-tuple of sequences of real numbers.

**Definition 1.3** (Vector Convergence). Let $x^{(n)}, x \in \mathbb{R}^d$. Then we say $x^{(n)}$ converges to $x$, written $x^{(n)} \to x$, if $\| x^{(n)} - x \| \to 0$.

The basic thing to understand here is that vector convergence is equivalent to convergence in each coordinate:

**Proposition 1.4.** Let $x^{(n)}, x \in \mathbb{R}^d$ where $x^{(n)} = (x^{(n)}_1, \ldots, x^{(n)}_d)$ and $x = (x_1, \ldots, x_d)$. Then $x^{(n)} \to x$ as vectors iff for each $i = 1, \ldots, d$ one has $x^{(n)}_i \to x_i$ as real numbers.
Proof. We have the sequence of inequalities for any fixed $i = 1, \ldots, d$:
\begin{equation}
|x_i^{(n)} - x_i| \leq \|x^{(n)} - x\| \leq \sqrt{d} \max_{j} |x_j^{(n)} - x_j|.
\end{equation}
In particular \(\|x^{(n)} - x\| \to 0\) implies \(|x_i^{(n)} - x_i| \to 0\) for each \(i = 1, \ldots, d\). And if \(|x_i^{(n)} - x_i| < \frac{\varepsilon}{\sqrt{d}}\) for \(n > N_i\) we have \(\|x^{(n)} - x\| < \varepsilon\) whenever \(n > N = \max\{N_1, \ldots, N_d\}\). \(\square\)

1.2. Bounded Sets. We begin with:

**Definition 1.5** (Boundedness). A subset \(S \subseteq \mathbb{R}^d\) is said to be **bounded** if there exists an \(M \geq 0\) with \(\|x\| \leq M\) for all \(x \in S\).

Note that a sequence \(S \subseteq \mathbb{R}^d\) is bounded iff it has no sequences \(x^{(n)} \in S\) with \(\|x^{(n)}\| \to \infty\). A fundamental fact about bounded sets is the following:

**Theorem 1.6** (Bolzano-Weierstrass Theorem). Let \(S \subseteq \mathbb{R}^d\) be a bounded set. Then every sequence \(x^{(n)} \in S\) has a convergent subsequence.

Proof. Suppose \(\|x\| \leq M\) for all \(x \in S\). Then by (1) we have \(|x_i^{(n)}| \leq M\) for all \(i = 1, \ldots, d\). By the Bolzano-Weierstrass theorem for \(\mathbb{R}\) there exists a subsequence \(x_1^{(n_k)} \to x_1\). Replacing \(x^{(n)}\) by \(x^{(n_k)}\) we can WLOG assume the original sequence was such that \(x_1^{(n_k)} \to x_1\). Now by a similar argument there exists another subsequence \(x^{(n_k)}\) with \(x_2^{(n_k)} \to x_2\). By inductively passing to further subsequences we can construct a grand subsequence \(x^{(n_k)}\) with \(x_i^{(n_k)} \to x_i\) for each \(i = 1, \ldots, d\). By Proposition 1.4 we have \(x^{(n_k)} \to x\). \(\square\)

2. Open, Closed, Compact, and Connected Sets

We now come to two fundamental notions of analysis in \(\mathbb{R}^d\).

**Definition 2.1**. A set \(O \subseteq \mathbb{R}^d\) is called **open** iff for each \(x \in O\) where exists an \(\epsilon = \epsilon(x) > 0\) such that \(y \in O\) for all \(\|x - y\| < \epsilon\).

As set \(C \subseteq \mathbb{R}^d\) is called **closed** iff for each sequence \(x^{(n)} \in C\), with \(x^{(n)} \to x\), we must have \(x \in C\).

In other words an open set is one in which at every point there is “a little bit of room” in the sense that one can always move away from any given point a small (fixed) distance in any direction and still remain in the set. On the other hand a closed set is one that “contains all its limits”.

By definition we let the empty set \(\emptyset \subseteq \mathbb{R}^d\) be both open and closed. In addition \(\mathbb{R}^d\) itself is both open and closed. It turns out these are the only subsets of \(\mathbb{R}^d\) which are both open and closed.

One thing that’s easy to show from these definitions is the following which often comes up in practice:

**Theorem 2.2**. The union of any collection of open sets is open, and the intersection of any collection of closed sets is closed.

The intersection of any finite collection of open sets is open, and the union of any finite collection of closed sets is closed.

Simple examples show that the finite condition for intersections of open sets and unions of closed sets in this theorem is necessary. For example fix \(a < b\) and consider infinite intersections of open intervals \(\cap_n (a - 1/n, b + 1/n) = [a, b]\), and infinite unions of closed intervals \(\cup_n [a + 1/n, b - 1/n] = (a, b]\).

Here is the most basic example of an open set:

**Example 2.1**. Any set of the form \(B_R(x) = \{y \in \mathbb{R}^d \mid \|x - y\| < R\}\) is an open set by the triangle inequality. If \(y \in B_R(x)\) set \(\epsilon = R - \|x - y\| > 0\). Then for all \(\|z - y\| < \epsilon\) we have
\[\|z - x\| \leq \|z - y\| + \|y - x\| < \epsilon + (R - \epsilon) = R,\]
Thus \(z \in B_R(x)\) as well.

Thus, any set \(O = \cup O_\alpha\) which is a union of open balls \(O_\alpha = B_{r_\alpha}(x_\alpha)\) is open. Conversely, by the definition every open set is the union sufficiently small open balls centered at each of its points.

We now give a few additional definitions which help to clarify the relationship between open and closed sets.
Definition 2.3. Let $S \subseteq \mathbb{R}^d$ be a set. We call a point $x \in S$ an interior point if there exists some $R > 0$ such that $B_R(x) \subseteq S$. We set $\text{Int}(S)$ to be the set of all interior points of $S$. This is called the interior of $S$.

Let $S \subseteq \mathbb{R}^d$ be a set and define

$$\text{Lim}(S) = \{ y \in \mathbb{R}^d \mid \text{there is a sequence } x^{(n)} \in S \text{ with } x^{(n)} \rightarrow y \}.$$ 

We call $\text{Lim}(S)$ the limit points of $S$.

Let $S \subseteq \mathbb{R}^d$ be a set. Then a point $y \in \mathbb{R}^d$ (not necessarily in $S$) is called a boundary point if for each $R > 0$ one has both $B_R(y) \cap S \neq \emptyset$ and $B_R(y) \cap S^c \neq \emptyset$. The set of all boundary points of $S$ is denoted by $\partial S$ and it called the boundary of $S$.

These notions are all tied together as follows:

Theorem 2.4. Let $S \subseteq \mathbb{R}^d$ be a set. Then the following hold:

i) One always has $\text{Int}(S) \subseteq S$ and $S$ is open iff $S = \text{Int}(S)$.

ii) One always has $S \subseteq \text{Lim}(S)$, and $S$ is closed iff $S = \text{Lim}(S)$.

iii) One has $\text{Int} (\text{Lim}(S)) = \text{Int}(S)$. In particular $\text{Int}(S)$ is always open.

iv) One has $\text{Lim}(\text{Lim}(S)) = \text{Lim}(S)$. In particular $\text{Lim}(S)$ is always closed.

v) One always has $\text{Int}(\text{Lim}(S)) = S \setminus \partial S$. In particular $S$ is open iff $\partial S \subseteq S^c$.

vi) One always has $\text{Lim}(S) = S \cup \partial S$. In particular $S$ is closed iff $\partial S \subseteq S$.

vii) One always has $\partial S = \partial S^c$.

The result can basically be summed up by saying two things: First is that a limit point of limit points is a limit point. Second is that any set can be decomposed into two disjoint pieces consisting of all interior points, and some of its boundary points. A set is open iff it contains none of its boundary points, and is closed iff it contains all of its boundary points. Notice that most sets fall somewhere in between these two extremes so they are neither open nor closed.

Note also that a particular consequence of v), vi), and vii) is that a set is open iff its complement is closed. This is because $\mathcal{O}$ is open iff $\partial \mathcal{O}^c = \partial \mathcal{O} \subseteq \mathcal{O}^c$. And $\mathcal{O}^c$ is closed iff $\partial \mathcal{O}^c \subseteq \mathcal{O}^c$.

Selected proofs. A number of the above proofs are essentially the definition, or follow at once from the definitions. For example i) and ii) are just the definitions of open and closed sets. Parts iii), v), and vii) are also immediate from definitions and Example 2.1 which helps to show the inclusion $\text{Int}(S) \subseteq \text{Int} (\text{Int}(S))$. Thus, the meat of the theorem is to show iv) and vi).

To see that $\text{Lim}(\text{Lim}(S)) = \text{Lim}(S)$ let $x^{(n)} \in \text{Lim}(S)$ be such that $x^{(n)} \rightarrow x$. For each $n$ choose a sequence $y^{(m),n} \in S$ with $y^{(m),n} \rightarrow x^{(n)}$. Next, for each $k \in \mathbb{N}$ choose indices $n_k$ and then $m_k$ with the properties $\| x^{(n_k)} - x \| < 1/k$, and then $\| y^{(m_k),n_k} - x^{(n_k)} \| < 1/k$. Then if $y^{(k)} = y^{n_k, (m_k)}$ we get $\| y^{(k)} - x \| < 2/k \rightarrow 0$ as $k \rightarrow \infty$. Thus $y^{(k)} \in S$ and $y^{(k)} \rightarrow x$ so $x \in \text{Lim}(S)$.

To show $\text{Lim}(S) = S \cup \partial S$ note first that $\partial S \subseteq \text{Lim}(S)$ is easy because of $x \in \partial S$ just choose $x^{(n)} \in S$ with $x^{(n)} \in B_{1/n}(x)$. Since $S \subseteq \text{Lim}(S)$ for free we get $S \cup \partial S \subseteq \text{Lim}(S)$.

On the other hand if $x \in \text{Lim}(S) \setminus S$ (assuming this is not empty) then let $x^{(n)} \in S$ with $x^{(n)} \rightarrow x$. Then for each $\epsilon > 0$ we have $B_{\epsilon}(x) \cap S \neq \emptyset$ because $x^{(n)} \in B_{\epsilon}(x)$ for all $n > N = N(\epsilon)$. And of course $x \notin S$ implies $B_{\epsilon}(x) \cap S^c \neq \emptyset$ as well. Thus $x \in \partial S$ showing $\text{Lim}(S) \setminus S \subseteq \partial S$. Thus $\text{Lim}(S) \subseteq S \cup \partial S$ and we are done.

2.1. Compact Sets. Next we introduce a notion which is a vast generalization of closed and bounded intervals $[a, b] \subset \mathbb{R}$.

Definition 2.5 (Compact sets). Let $\mathcal{C} \subseteq \mathbb{R}^d$. Then we say $\mathcal{C}$ is compact iff for each sequence $x^{(n)} \in \mathcal{C}$ there exists a convergent subsequence $x^{(n_k)}$, and $x^{(n_k)} \rightarrow y \in \mathcal{C}$.

The important thing is that its easy to identity compact sets.

Theorem 2.6 (Heine-Borel Theorem). A set $\mathcal{C} \subseteq \mathbb{R}^d$ is compact iff it is both closed and bounded.

Proof. If $\mathcal{C}$ is compact let $x^{(n)} \in \mathcal{C}$ with $x^{(n)} \rightarrow x$. Then there exists a subsequence $x^{(n_k)}$ and a point $y \in \mathcal{C}$ with $x^{(n_k)} \rightarrow y$. But then $y = x$ because any sequence of a convergent sequence must converge to the same limit. This shows $\mathcal{C}$ is closed. In addition if $x^{(n)} \in \mathcal{C}$ with $\| x^{(n)} \| \rightarrow \infty$, we’d have $\| x^{(n_k)} \| \rightarrow \infty$ for
every subsequence as well. This contradicts that \( x^{(n)} \) must have at least one convergent subsequence. Thus \( C \) must be bounded.

Now assume \( C \) is both closed about bounded. By Theorem 1.6 if \( x^{(n)} \in C \) is any sequence it must have a convergent subsequence \( x^{(n_k)} \to x \). But since \( C \) is closed we must have \( x \in C \) as well. Thus \( C \) is compact. □

2.2. Connected Sets. We now come to one final notion here which is the appropriate vector generalization of an interval \( I \subseteq \mathbb{R} \).

**Definition 2.7 (Connected Sets).** A set \( S \subseteq \mathbb{R} \) is said to be disconnected if there exist two nonempty disjoint subsets \( A, B \subseteq S \) such that \( S = A \cup B \), and \( \partial A \cap \partial B \cap S = \emptyset \).

A set \( S \subseteq \mathbb{R} \) is called connected if it is not disconnected.

In other words a disconnected set is one which can be written as a nonempty disjoint union \( S = A \cup B \) in such a way that any common boundary point to both \( A \) and \( B \) must lie outside of \( S \). Intuitively speaking this means that \( S \) is separated into two nonempty disjoint pieces in such a way that one cannot pass from one to the other without first leaving \( S \). Another way to think about this condition is that if \( S \) is connected, and we take some nontrivial decomposition \( S = A \cup B \), then at least one point of \( A \) or \( B \) can be reached by a sequence from the other set. To be precise:

**Theorem 2.8.** A set \( S \subseteq \mathbb{R}^d \) is connected iff whenever \( S = A \cup B \) with both \( A, B \) disjoint and nonempty, at least one of the following two options hold:

i) There exists a sequence \( x^{(n)} \in A \) with \( x^{(n)} \to x \in B \).

ii) There exists a sequence \( y^{(n)} \in B \) with \( y^{(n)} \to y \in A \).

**Proof.** Note that a set is connected iff whenever \( S = A \cup B \) is a nontrivial disjoint union \( S \) has at least one common boundary point of \( A \) and \( B \). Let \( x \in S \cap \partial A \cap \partial B \) be that common point. Then \( x \in A \) or \( x \in B \). In the first case since \( x \in \partial B \) we have by part vi) of Theorem 2.4 that there exists \( y^{(n)} \in B \) with \( y^{(n)} \to x \). On the other hand if \( x \in B \) we also have \( x \in \partial A \), so again by by vi) of Theorem 2.4 there exists a sequence \( x^{(n)} \in A \) with \( x^{(n)} \to x \). □

Finally, its worth pointing out that being connected is equivalent to being an interval when \( d = 1 \).

**Definition 2.9.** A set \( I \subseteq \mathbb{R} \) is called an interval if for each \( a < b \) with \( a, b \in I \) one has \( \xi \in I \) for all \( a < \xi < b \).

**Theorem 2.10.** A subset \( S \subseteq \mathbb{R} \) is connected if it is an interval.

**Proof.** Suppose \( S \subseteq \mathbb{R} \) is a nonempty set with more than two points and let \( a < b \) with \( a, b \in S \). If \( S \) were not an interval then there exists a \( \xi \) with \( a < \xi < b \) and \( \xi \notin S \). Then \( A = (-\infty, \xi) \cap S \) and \( B = (\xi, \infty) \cap S \) forms a nontrivial decomposition of \( S \) into disjoint sets which have the property that no sequence in \( A \) can converge to a point of \( B \) and vice versa. Thus \( S \) cannot be connected by the previous theorem. Put another way intervals are the only possible candidates for connected sets in \( \mathbb{R} \).

On the other hand let \( I \subseteq \mathbb{R} \) be any bounded closed interval, say \( I = [a, b] \), and let \( I = A \cup B \) with neither \( A \) nor \( B \) empty and \( A \cap B = \emptyset \). WLOG assume \( b \in \sup(B) \). Let \( x = \sup(A) \leq b \). If \( x = b \) there would be a sequence \( x^{(n)} \in A \) with \( x^{(n)} \to x \in B \). On the other hand if \( x < b \) then \( (x, b] \subseteq B \), so there exists a sequence \( y^{(n)} \in B \) with \( y^{(n)} \to x \in A \). This shows \( I \) is connected by Theorem 2.8, and so closed and bounded intervals are always connected.

Now suppose \( I \) is any interval, not necessarily closed or bounded. Suppose we could write \( I = A \cup B \) as a nontrivial disjoint union with \( \partial A \cap \partial B \cap I = \emptyset \). Choose some \( a \in A \) and \( b \in B \) and WLOG assume \( a < b \). Since \( I \) is an interval \( [a, b] \subseteq I \). But then we would have \( [a, b] = (A \cap [a, b]) \cup (B \cap [a, b]) \) is a nontrivial disjoint union with \( \partial (A \cap [a, b]) \cap \partial (B \cap [a, b]) \cap [a, b] \subseteq \partial A \cap \partial B \cap I = \emptyset \), which we just showed is impossible. Thus any interval must be connected. □

3. Continuous Functions

In the final part of these notes we discuss the notion of continuous vector valued functions on subsets of \( \mathbb{R}^d \).
Definition 3.1. Let $S \subseteq \mathbb{R}^d$ be given and $f : S \to \mathbb{R}^k$. We say $f$ is continuous at $x \in S$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for each $y \in S$ with $\|x - y\| < \delta$ one has $\|f(x) - f(y)\| < \epsilon$.

A function $f : S \to \mathbb{R}^k$ is called continuous on $S$ if it is continuous at each point of $S$.

Just as in the single variable case there is also a sequential definition of continuity. In addition to this there is a version that can be stated solely in terms of open sets. This is given by the following:

Theorem 3.2 (Equivalent definitions of continuity). Let $S \subseteq \mathbb{R}^d$ be given and $f : S \to \mathbb{R}^k$, and fix a point $x \in S$. Then the following conditions are equivalent:

i) $f$ is continuous at $x$.

ii) For every sequence $x^{(n)} \in S$ with $x^{(n)} \to x$ one has $f(x^{(n)}) \to f(x)$.

iii) For every open set $U \subseteq \mathbb{R}^k$ with $f(x) \in U$ there exists an open set $V \subseteq \mathbb{R}^d$ with $x \in V \cap S \subseteq f^{-1}(U)$.

In addition $f$ is continuous on $S$ iff for each open set $U \subseteq \mathbb{R}^k$ there exists an open set $V \subseteq \mathbb{R}^d$ with $V \cap S = f^{-1}(U)$. The same result holds if one replaces $U,V$ by closed sets.

Proof. Note that the definition of continuity can be written as: given $\epsilon > 0$ there exists $\delta > 0$ such that $f(B_\delta(x) \cap S) \subseteq B_\epsilon(f(x))$. Thus i)⇔iii) is immediate from the fact that open balls are open sets, and every open set contains an open ball about each of its points.

To show i)⇒ ii) let $x^{(n)} \to x$ in $S$ and let $\epsilon > 0$ be given. Let $\delta = \delta(\epsilon) > 0$ be given in the definition of continuity of $f$ at $x$, and choose $N = N(\delta)$ in the definition of convergence of $x^{(n)} \to x$. Then $\|x^{(n)} - x\| < \delta$ for all $n > N$, and since $f(B_\delta(x) \cap S) \subseteq B_\epsilon(f(x))$ we have $\|f(x^{(n)}) - f(x)\| < \epsilon$ for all $n > N$. Thus $f(x^{(n)}) \to f(x)$.

To show ii)⇒ i) we prove the contrapositive. If $f$ were not continuous at $x$ then there would exist a fixed $\epsilon_0 > 0$ such that for any $\delta > 0$ one could find at least one point $x_\delta \in S$ with $\|x - x_\delta\| < \delta$ but $\|f(x) - f(x_\delta)\| \geq \epsilon_0$. Doing this for the sequence $\delta_n = 1/n$, and setting $x_n = x_{\delta_n}$, we have produced a sequence of vectors $x_n \in S$ with $x_n \to x$ but $\|f(x) - f(x_n)\| \geq \epsilon_0 > 0$. Thus $f(x_n) \to f(x)$ is prohibited for this sequence.

The final claim of the theorem follows by combining i) and iii) the theorem, and using Theorem 2.2 and the duality between open and closed sets. The details of this are left to the reader.

We now state the two fundamental theorems about continuous functions on $\mathbb{R}^d$. The first is:

Theorem 3.3 (Generalized Min/Max Theorem). Let $C \subseteq \mathbb{R}^d$ be a compact set and suppose $f : C \to \mathbb{R}^k$ is continuous. Then $f(C)$ is compact.

In the particular case when $k = 1$ there exists points $x_{\min}, x_{\max} \in C$ such that:

$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

for all $x \in C$.

Proof. Let $y^{(n)} \in f(C)$ be any sequence, and choose $x^{(n)} \in C$ with $f(x^{(n)}) = y^{(n)}$. Then there exists a convergent subsequence with $x^{(n_k)} \to x \in C$. By continuity $y^{(n_k)} \to y = f(x) \in f(C)$. Thus every sequence in $f(C)$ has a subsequence converging to a point of $f(C)$, so $f(C)$ is a compact set.

The second main result is:

Theorem 3.4 (Generalized Intermediate Value Theorem). Let $S \subseteq \mathbb{R}^d$ be a connected set and suppose $f : S \to \mathbb{R}^k$ is continuous. Then $f(S)$ is connected.

In the particular case when $k = 1$, if there exists $x,y \in S$ and $\xi \in \mathbb{R}$ with $f(x) < \xi < f(y)$, then there exists some $z \in S$ with $f(z) = \xi$.

Proof. Let $f(S) = A \cup B$ where both $A,B$ are nonempty and $A \cap B = \emptyset$. Then $S = f^{-1}(A) \cup f^{-1}(B)$ is a nontrivial decomposition of $S$ into disjoint sets. Suppose WLOG that case i) of Theorem 2.8 holds for $f^{-1}(A)$ and $f^{-1}(B)$. Then there exists some $x^{(n)} \in f^{-1}(A)$ with $x^{(n)} \to x \in f^{-1}(B)$. Then $f(x^{(n)}) \in A$, and by continuity $f(x^{(n)}) \to f(x) \in B$. Thus $f(S)$ is connected by Theorem 2.8.
