NOTES ON FOURIER SERIES

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1. Trig Identities and Integrals

Recall that the matrix:

\[ R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \]

represents the linear transformation of \( \mathbb{R}^2 \) which is counterclockwise rotation through \( \theta \) degrees. In particular one has the group identity \( R_A R_B = R_{A+B} \). Writing out the matrix elements on both sides of this identity yields the standard trig identities:

\[
\begin{align*}
\cos(A + B) &= \cos(A) \cos(B) - \sin(A) \sin(B), \\
\sin(A + B) &= \sin(A) \cos(B) + \cos(A) \sin(B).
\end{align*}
\]

Using the fact that cosine is even while sine is odd we can average these identities with \( A \) replaced by \( \pm A \) or \( B \) replaced by \( \pm B \) we have:

\[
\begin{align*}
\cos(A) \cos(B) &= \frac{1}{2} (\cos(A + B) + \cos(A - B)), \\
\sin(A) \sin(B) &= \frac{1}{2} (\cos(A - B) - \cos(A + B)), \\
\sin(A) \cos(B) &= \frac{1}{2} (\sin(A + B) + \sin(A - B)).
\end{align*}
\]

Integrating these identities over \([0, \pi]\) for \( A = nx \) and \( B = mx \), where \( n, m \in \mathbb{Z} \) we arrive at the following orthogonality relations for trig functions:

\[
\begin{align*}
(1a) \quad \int_0^{2\pi} \cos(nx) \cos(mx) dx &= \begin{cases} 
2\pi, & n = m = 0; \\
\pi, & n = m \neq 0; \\
0, & n \neq m.
\end{cases} \\
(1b) \quad \int_0^{2\pi} \sin(nx) \sin(mx) dx &= \begin{cases} 
0, & n = m = 0; \\
\pi, & n = m \neq 0; \\
0, & n \neq m.
\end{cases} \\
(1c) \quad \int_0^{2\pi} \sin(nx) \cos(mx) dx &= 0, \quad \text{all} \quad n, m.
\end{align*}
\]
2. Fourier Coefficients

Let \( f : [0, 2\pi] \to \mathbb{R} \) be a Riemann integrable function. Then \( fg \) is Riemann integrable for any continuous function \( g : [0, 2\pi] \to \mathbb{R} \). In particular we can define the coefficients:

\[
a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx, \quad a_n = \frac{1}{\pi} \int_0^{2\pi} \cos(nx)f(x)dx, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} \sin(nx)f(x)dx,
\]

where the last two are defined for \( n \geq 1 \). From this information we set:

\[
S_N(f)(x) = \sum_{n=0}^{N} \left( a_n \cos(nx) + b_n \sin(nx) \right),
\]

where we set \( b_0 = 0 \) for convenience. This is the partial Fourier series associated to \( f \). The key question we are after here is in what sense can one have convergence \( S_N(f) \to f \) as \( N \to \infty \).

First of all assume \( f(x) = \sum_{n=0}^{N_0} (A_n \cos(nx) + B_n \sin(nx)) \). In other words \( f \) is a-priori a finite Fourier series. Then the orthogonality relations \( (1) \) show that:

\[
a_n = \begin{cases} A_n, & 0 \leq n \leq N_0; \\ 0, & n > N_0, \end{cases}, \quad b_n = \begin{cases} B_n, & 1 \leq n \leq N_0; \\ 0, & n > N_0. \end{cases}
\]

In particular \( S_N(f) = f \) whenever \( N \geq N_0 \) for finite Fourier series. At least this shows our definition of the coefficients \((a_n, b_n)\) makes sense.

3. The Dirichlet Kernel

When doing certain computations it is convenient to replace the sum version of \( S_N(f) \) with an integral version. To do this we simply expand the formulas for \((a_n, b_n)\) inside the formula for \( S_N(f) \). This gives:

\[
S_N(f)(x) = a_0 + \sum_{n=1}^{N_0} (a_n \cos(nx) + b_n \sin(nx)) = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx + \sum_{n=1}^{N_0} \frac{1}{\pi} \int_0^{2\pi} \cos(nx) \cos(ny)f(y)dy + \sum_{n=1}^{N_0} \frac{1}{\pi} \int_0^{2\pi} \sin(nx) \sin(ny)f(y)dy \\
= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} f(x)dx + \sum_{n=1}^{N_0} \left( \cos(nx) \cos(ny) + \sin(nx) \sin(ny) \right)dy.
\]

Using the trig identity \( \cos(A) \cos(B) + \sin(A) \sin(B) = \cos(A - B) \) the function inside the integral becomes:

\[
S_N(f)(x) = \int_0^{2\pi} D_N(x-y)f(y)dy, \quad \text{where} \quad D_N(\theta) = \frac{1}{\pi} \left( \frac{1}{2} + \sum_{n=1}^{N} \cos(n\theta) \right).
\]

The function \( D_N(\theta) \) is known as the Dirichlet kernel. Note that the formula we gave still involves a sum, but this is rectified by again using trig identities:

\[
2\pi \sin \left( \frac{1}{2} \theta \right) D_N(\theta) = \sin \left( \frac{1}{2} \theta \right) + 2 \sum_{n=1}^{N} \sin \left( \frac{1}{2} \theta \right) \cos(n\theta),
\]

\[
= \sin \left( \frac{1}{2} \theta \right) + \sum_{n=1}^{N} \left[ \sin \left( (n + \frac{1}{2})\theta \right) - \sin \left( (n - \frac{1}{2})\theta \right) \right],
\]

\[
= \sin \left( (N + \frac{1}{2})\theta \right),
\]

where the last line follows from the previous by noting that the sum is telescoping. Therefore we have the explicit formula:

\[
S_N(f)(x) = \int_0^{2\pi} D_N(x-y)f(y)dy, \quad \text{where} \quad D_N(\theta) = \frac{\sin \left( (N + \frac{1}{2})\theta \right)}{2\pi \sin \left( \frac{1}{2} \theta \right)}.
\]
We'll end this section with a few more observations about the Dirichlet kernel and partial sums $S_N(f)$. These are largely based on the following lemma:

**Lemma 3.1.** Let $G : \mathbb{R} \to \mathbb{R}$ be a $2\pi$-periodic function (that is $G(x) = G(x + 2\pi)$ for all $x \in \mathbb{R}$) which is Riemann integrable in the sense that $G : [0, 2\pi] \to \mathbb{R}$ is Riemann integrable. Then $G$ is Riemann integrable on any interval $[a, b]$ and one has the formula

$$\int_a^{a + 2\pi} G(x) \, dx = \int_0^{2\pi} G(x) \, dx$$

for any $a \in \mathbb{R}$.

**Proof.** The integrability is easy because $[a, b] \subseteq [-2N\pi, 2N\pi]$ for sufficiently large $N$, and the integrability of $G$ on any interval of the form $[2n\pi, 2m\pi]$ for integers $n < m$ follows by periodicity and inductively applying Theorem 33.6 of the text.

To get the period formula for the integral is suffices to assume $0 \leq a < 2\pi$ (otherwise use $G(x) = G(x + 2k\pi)$ to shift things back). Now:

$$\int_a^{a + 2\pi} G(x) \, dx = \int_a^{2\pi} G(x) \, dx + \int_{2\pi}^{a + 2\pi} G(x) \, dx = \int_a^{2\pi} G(x) \, dx + \int_0^a G(x + 2\pi) \, dx = \int_0^{2\pi} G(x) \, dx$$

where we have used Theorems 33.6 and 34.4 of the text and $G(x + 2\pi) = G(x)$.

From this lemma we have:

**Lemma 3.2.** Let $f : [0, 2\pi] \to \mathbb{R}$ be Riemann integrable. Then we have the following identities for the partial sums $S_N(f)$:

(2a) \hspace{1cm} S_N(f)(x) = \int_{-\pi}^{\pi} D_N(\theta) f(x - \theta) d\theta = \int_{-\pi}^{\pi} D_N(\theta) f(x + \theta) d\theta ,

(2b) \hspace{1cm} S_N(f)(x) - f(x) = \int_{-\pi}^{\pi} D_N(\theta) (f(x - \theta) - f(x)) d\theta = \int_{-\pi}^{\pi} D_N(\theta) (f(x + \theta) - f(x)) d\theta ,

(2c) \hspace{1cm} S_N(f)(x) - L = 2 \int_0^{\pi} D_N(\theta) \left( \frac{f(x + \theta) + f(x - \theta)}{2} - L \right) d\theta , \quad \text{for any } L \in \mathbb{R} .

**Proof.** The first identity on line (2a) follows because $\int_0^{2\pi} D_N(x - y) f(y) \, dy = \int_{-\pi}^{\pi} D_N(\theta) f(x - \theta) d\theta$ after the change of variables $x - y = \theta$, and then shifting the integral to $[-\pi, \pi]$.

The second identity on line (2a) follows from the first and the fact that $D_N(\theta)$ is even. The identities on lines (2b) follow from (2a) and the fact that $\int_{-\pi}^{\pi} S_N(\theta) d\theta = 1$. Identity (2c) follows by averaging the two identities on line (2b) and then using the fact that both $D_N(\theta)$ and $\frac{f(x + \theta) + f(x - \theta)}{2}$ are even functions. \qed

4. **Uniform Convergence**

We now state a basic convergence result for Fourier series. For this it helps to record a simple calculation which often comes up.

**Lemma 4.1.** There exists a constant $A$ such that:

$$2 \leq \frac{\theta}{\sin(\frac{1}{2}\theta)} \leq A , \quad \text{for all } 0 \leq \theta \leq \pi .$$

**Proof.** The lower bound follows because $|\sin(\theta)| \leq |\theta|$ for all $\theta$ by the mean value theorem. The upper bound follows because $\sin(\frac{1}{2}\theta)$ is continuous on $[0, \pi]$ and only vanishes at $\theta = 0$ where where know the fraction has a limit (by L’Hospital’s rule). Thus $\frac{\theta}{\sin(\frac{1}{2}\theta)}$ is continuous on $[0, \pi]$ and therefore has a maximum. \qed

**Theorem 4.2.** Let $f : \mathbb{R} \to \mathbb{R}$ be a $2\pi$-periodic function such that $f'(x)$ exists for all $x \in \mathbb{R}$ and such that $f' : [0, 2\pi] \to \mathbb{R}$ is Riemann integrable. Then there exists a uniform constant $C > 0$ independent of $f$ and $N$ such that:

(3) \hspace{1cm} |S_N(f)(x) - f(x)| \leq C \frac{\ln(N)}{N} \sup_y |f'(y)| , \quad \text{for all } x \in \mathbb{R} .

In particular $S_N(f) \to f$ uniformly on $\mathbb{R}$.
By the previous Lemma and the mean value theorem we have:

\[ I_1 = \int_{-\frac{\pi}{N}}^{\frac{\pi}{N}} D_N(\theta)(f(x - \theta) - f(x))d\theta, \quad I_2 = \int_{-\frac{\pi}{N}}^{\frac{\pi}{N}} D_N(\theta)(f(x - \theta) - f(x))d\theta, \]

and where the integral \( I_3 \) is between \([-\pi, -\frac{1}{N}]\). The integrals \( I_2 \) and \( I_3 \) are essentially symmetric and will be treated by the same argument.

To bound \( I_1 \) we simply write:

\[ |I_1| \leq \int_{-\frac{\pi}{N}}^{\frac{\pi}{N}} |\theta D_N(\theta)| \cdot \left| \frac{f(x - \theta) - f(x)}{\theta} \right| d\theta \]

Now \( |\theta D_N(\theta)| \leq \frac{1}{2\pi} A \) by the previous Lemma. It is important to note that the constant \( A \) here does not depend on \( N \). On the other hand by differentiability \( \frac{f(x - \theta) - f(x)}{\theta} \) is also continuous, and by the mean value theorem we have the bound \( \left| \frac{f(x - \theta) - f(x)}{\theta} \right| \leq \sup_y |f'(y)| \). Combining all this we get:

\[ \text{(4)} \quad |I_1| \leq C_1 \frac{1}{N} \sup_y |f'(y)|, \]

where \( C_1 \) is a universal constant not depending on \( f \) or \( N \).

To bound \( I_2 \) we need to integrate by parts one time. Doing this yields the identity \( I_2 = J_1 + J_2 \) where:

\[ J_1 = - \frac{1}{2\pi(N + \frac{1}{2})} \cos \left((N + \frac{1}{2})\theta\right) \left( \frac{f(x - \theta) - f(x)}{\sin \left(\frac{1}{2}\theta\right)} \right) \bigg|_{\theta = \pi}^{\theta = \frac{\pi}{N}}, \]

\[ J_2 = \frac{1}{2\pi(N + \frac{1}{2})} \int_{\frac{\pi}{N}}^{\pi} \cos \left((N + \frac{1}{2})\theta\right) \frac{d}{d\theta} \left( \frac{f(x - \theta) - f(x)}{\sin \left(\frac{1}{2}\theta\right)} \right) d\theta. \]

By the previous Lemma and the mean value theorem we have:

\[ \left| \frac{f(x - \theta) - f(x)}{\sin \left(\frac{1}{2}\theta\right)} \right| \leq A \sup_y |f'(y)|. \]

In particular:

\[ \text{(5)} \quad |J_1| \leq C_2 \frac{1}{N} \sup_y |f'(y)|, \]

where \( C_2 \) is a universal constant not depending on \( f \) or \( N \).

Likewise a similar application of the mean value theorem and the previous lemma also gives:

\[ \left| \frac{d}{d\theta} \left( \frac{f(x - \theta) - f(x)}{\sin \left(\frac{1}{2}\theta\right)} \right) \right| \leq A \left( A + \frac{1}{2} A^2 \right) \sup_y |f'(y)|. \]

Integrating this bound over \([\frac{1}{N}, \pi]\) gives us:

\[ \text{(6)} \quad |J_2| \leq C_3 \frac{\ln(N)}{N} \sup_y |f'(y)|. \]

Combining (4), (5), and (6) gives (3). 

\[ \square \]

**5. Approximation Theorems for Integrable Functions**

Our next goal is to discuss more general convergence theorems for Fourier series. It turns out these have less to do with proving additional things about the Dirichlet kernel, and more to do with general theorems about how one can approximate integrable functions. The first such lemma is the following:

**Lemma 5.1.** Let \( f : [a, b] \to \mathbb{R} \) be a Riemann integrable function with \( |f| \leq M \). Then for each \( \epsilon > 0 \) there exists a step function \( \varphi \) with \( |\varphi| \leq M \) and \( \int_a^b |f - \varphi|dx < \epsilon \).
Proof. Let $\epsilon > 0$ and choose a partition $P = \{a = t_0 < \ldots < t_k = b\}$ with:
\[
\sum_{k=1}^{n} (\sup f - \inf f)(t_k - t_{k-1}) < \epsilon ,
\]
where each sup and inf is taken on $[t_{k-1}, t_k]$. Next, choose some collection of points $t_{k-1} \leq x_k \leq t_k$ and define the step function:
\[
\varphi(x) = f(x_k) , \quad \text{when} \quad t_{k-1} < x < t_k ,
\]
and $\varphi(x) = 0$ at the endpoints $x = t_k$ for each $k = 0, \ldots, n$ (the precise definition of $\varphi$ at these points is of course irrelevant). Then:
\[
\int_{a}^{b} |f - \varphi|dx \leq \sum_{k=1}^{n} \sup_{x \in [t_{k-1}, t_k]} |f(x) - f(k)| (t_k - t_{k-1}) \leq \sum_{k=1}^{n} (\sup f - \inf f)(t_k - t_{k-1}) < \epsilon ,
\]
as was to be shown. \qed

The next step is to show one can replace step functions with smooth functions. The technical part boils down to the following lemma:

**Lemma 5.2.** Let $[a, b] \subseteq \mathbb{R}$ and $0 < \epsilon < \frac{b-a}{2}$. Then there exists an infinitely differentiable function $\varphi_\epsilon : \mathbb{R} \to \mathbb{R}$ such that:

i) $0 \leq \varphi_\epsilon(x) \leq 1$ for all $x \in \mathbb{R}$.

ii) $\varphi_\epsilon(x) = 0$ for all $x < a$ and $x > b$.

iii) $\varphi_\epsilon(x) = 1$ for all $a + \epsilon \leq x \leq b - \epsilon$.

**Proof.** Let $h(x)$ be an infinitely differentiable function with $0 \leq h \leq 1$ and $h(x) = 0$ for all $x \leq 0$ while $h(x) > 0$ otherwise. For example $h(x) = e^{-1/x}$ for $x > 0$ and zero otherwise works. Now define:
\[
H(x) = \frac{h(x)}{h(x) + h(1-x)} .
\]
Then $H$ is infinitely differentiable with $0 \leq H \leq 1$, $H(x) = 0$ when $x \leq 0$, and $H(x) \geq 1$ when $x \geq 1$. Finally define:
\[
\varphi_\epsilon(x) = H(e^{-1}(x-a)) H(e^{-1}(b-x)) .
\]
\qed

Now it is easy to show:

**Lemma 5.3.** Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function with $|f| \leq M$. Then for each $\epsilon > 0$ there exists a an infinitely differentiable function $\varphi$, which vanishes outside of $[a,b]$, and such that both $|\varphi| \leq M$ and $\int_{a}^{b} |f - \varphi|dx < \epsilon$.

**Proof.** By the version of this lemma for step functions and the triangle inequality it suffices to assume that $f$ is a step function. In other words we may assume there is a partition $P = \{a = t_0 < \ldots < t_k = b\}$ and:
\[
f(x) = \sum_{k=1}^{n} c_k 1_{(t_{k-1}, t_k)}(x) ,
\]
where $c_k$ are some constants and $1_{S}(x)$ is the function where $1_{S}(x) = 1$ for $x \in S$ and $1_{S}(x) = 0$ for $x \notin S$. For a small $\epsilon_0 > 0$ let:
\[
\varphi(x) = \sum_{k=1}^{n} c_k \varphi_{\epsilon_0, (t_{k-1}, t_k)}(x) ,
\]
where each $\varphi_{\epsilon_0, (t_{k-1}, t_k)}$ is the cutoff function associated with the interval $[t_{k-1}, t_k]$ in the previous lemma at scale $\epsilon_0$. Then we have:
\[
\int_{a}^{b} |f - \varphi|dx \leq \sum_{k=1}^{n} |c_k| \int |1 - \varphi_{\epsilon_0, (t_{k-1}, t_k)}|dx \leq 2n M \epsilon_0 ,
\]
where $M = \max_{k=1,\ldots,n} |c_k|$. Since both $n, M$ are fixed by $f$, we can choose $\epsilon_0 = \frac{\epsilon}{nM}$ (assume $M \neq 0$). Then $\int_{a}^{b} |f - \varphi|dx < \epsilon$ as desired. \qed
A particular consequence of this is the following:

**Theorem 5.4.** Let \( f : [0, 2\pi] \to \mathbb{R} \) be a Riemann integrable function. Then for each \( \epsilon > 0 \) there exists a \( 2\pi \)-periodic function \( \varphi : \mathbb{R} \to \mathbb{R} \) such that \( \varphi \) exists and is continuous, and one has \( \int_0^{2\pi} |f - \varphi|^2 \, dx < \epsilon \).

**Proof.** Let \( |f| \leq M \), and assume \( M \neq 0 \). Choose \( \varphi \) as in the previous lemma with \( \epsilon \) replaced by \( \frac{\epsilon}{2M} \). Then \( |f - \varphi|^2 \leq \left( |f| + |\varphi| \right) |f - \varphi| \leq 2M|f - \varphi| \). Upon integration we get the desired result. \( \square \)

### 6. Mean Square Convergence

We now return to the issue of convergence of Fourier series. The main notion we want to look at now is the following.

**Definition 6.1 (Mean Square Convergence).** Let \( f_n, f : [a, b] \to \mathbb{R} \) be a collection of Riemann integrable functions. Then we say \( f_n \to f \) in the mean square sense if:

\[
\int_a^b |f_n - f|^2 \, dx \to 0 , \quad \text{as} \quad n \to \infty .
\]

The material of the previous to sections tells us that:

- If \( f : \mathbb{R} \to \mathbb{R} \) is \( 2\pi \)-periodic function such that \( f'(x) \) exists for all \( x \in \mathbb{R} \) and such that \( f' : [0, 2\pi] \to \mathbb{R} \) is Riemann integrable, then \( S_N(f) \to f \) in the mean square sense on \( [0, 2\pi] \). This is simply because we have the much stronger result that \( |S_N(f) - f|^2 \to 0 \) uniformly.

- If \( f : [0, 2\pi] \to \mathbb{R} \) is Riemann integrable, then there exists a sequence of functions of continuously differentiable \( 2\pi \) periodic \( f_n : \mathbb{R} \to \mathbb{R} \) such that \( f_n \to f \) in the mean square sense.

Combining these two it stands to reason that if \( f : [0, 2\pi] \to \mathbb{R} \) is Riemann integrable then \( S_N(f) \to f \) in the mean square sense as well because we can write:

\[
S_N(f) - f = (S_N(f_n) - f_n) + (f_n - f) + S_N(f_n - f) ,
\]

where we at least know the first and second terms are small in the mean square sense by taking first \( n \) then \( N \) sufficiently large. It turns out the last term in controlled by the second term for free. This is because:

**Lemma 6.2 (Plancherel’s Inequality).** Let \( g : [0, 2\pi] \to \mathbb{R} \) be Riemann integrable. Then one has:

\[
(7) \quad \int_0^{2\pi} |S_N(g)|^2 \, dx \leq \int_0^{2\pi} |g|^2 \, dx .
\]

**Proof.** Expanding out we have:

\[
0 \leq \int_0^{2\pi} |S_N(g) - g|^2 \, dx \leq \int_0^{2\pi} |S_N(g)|^2 \, dx - 2 \int_0^{2\pi} S_N(g)g \, dx + \int_0^{2\pi} |g|^2 \, dx .
\]

Thus, it suffices to show the identity:

\[
\int_0^{2\pi} S_N(g)g \, dx = \int_0^{2\pi} |S_N(g)|^2 \, dx .
\]

This is a straightforward calculation writing \( S_N(g)(x) = \sum_{n=0}^{N} (a_n \cos(nx) + b_n \sin(nx)) \) and using the definition of the Fourier coefficients \( (a_n, b_n) \) and the integrated trig identities (1). Both sides are equal to the quantity \( 2\pi a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \). Further details are left to the reader. \( \square \)

Combining all of this we have:

**Theorem 6.3.** Let \( f : [0, 2\pi] \to \mathbb{R} \) be Riemann integrable. Then one has \( S_N(f) \to f \) in the mean square sense.

**Proof.** For numbers \( a, b \) we have \( |a + b|^2 \leq 2(a^2 + b^2) \). Thus:

\[
\int_0^{2\pi} |S_N(f) - f|^2 \, dx \leq 2 \int_0^{2\pi} |S_N(\varphi) - \varphi|^2 \, dx + 4 \int_0^{2\pi} |\varphi - f|^2 \, dx + 4 \int_0^{2\pi} |S_N(\varphi - f)|^2 \, dx ,
\]

\[
\leq 2 \int_0^{2\pi} |S_N(\varphi) - \varphi|^2 \, dx + 8 \int_0^{2\pi} |\varphi - f|^2 \, dx ,
\]

\[
\leq 2 \int_0^{2\pi} |S_N(\varphi)|^2 \, dx + 8 \int_0^{2\pi} |\varphi - f|^2 \, dx ,
\]

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where \( \varphi : [0, 2\pi] \to \mathbb{R} \) is some other integrable function which we can assume is \( 2\pi \)-periodic and continuously differentiable. First choose \( \varphi \) so that \( \int_{0}^{2\pi} |\varphi - f|^2 \, dx < \epsilon/16 \), then choose \( N_0 \) so large that \( N > N_0 \) implies \( \int_{0}^{2\pi} |S_N(\varphi) - \varphi|^2 \, dx < \epsilon/4 \). Then adding these we get \( \int_{0}^{2\pi} |S_N(f) - f|^2 \, dx < \epsilon \) for all \( N > N_0 \) so we are done. \( \square \)

7. Convergence at Jumps and the Gibbs Phenomenon

Our final topic of discussion here will be what happens at points where \( f(x) \) is discontinuous if we assume these are isolated points and \( f \) is otherwise a very nice differentiable function. Notice that this is relevant even in the case where \( f : [0, 2\pi] \to \mathbb{R} \) is a polynomial because in general we will not have \( f(0) = f(2\pi) \), so viewed as a periodic function \( f \) has a jump discontinuity at the endpoints.

The first result here tells us what we should expect the Fourier series to do at jump points. We can state this in a general way:

**Theorem 7.4.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a \( 2\pi \)-periodic function and fix some \( x \in \mathbb{R} \). Suppose there exists a number \( L \in \mathbb{R} \) such that \( h(\theta) = \frac{1}{\theta} (f(x+\theta) + f(x-\theta) - 2L) \) is integrable on the interval \([0, \pi]\). Then one has \( S_N(f)(x) \to L \).

**Remark 7.2.** A few remarks on this result are in order. First, the convergence \( S_N(f)(x) \to L \) is not uniform in \( x \) as we shall see in a moment. However, this result does imply certain pointwise convergence theorems for Fourier series.

Second, if \( f : [0, 2\pi] \to \mathbb{R} \) is merely integrable and \( f(x) \equiv 0 \) for \( 0 < a < x < b < 2\pi \), then \( S_N(f) \to 0 \) for all \( x \in (a, b) \) (and in fact one can show this convergence is uniform). This is Riemann’s principal of localization which says that if two integrable functions agree on an interval \( (a, b) \subset (0, 2\pi) \) and one of their Fourier series converges on that interval, then the other Fourier series must also converge on that interval, then it converges to the same value as the first.

**Proof.** Using formula (2c) we can write:

\[
S_N(f)(x) - L = 2 \int_{0}^{\pi} \sin((N + \frac{1}{2})\theta) h(\theta) \, d\theta , \quad \text{where} \quad h(\theta) = \frac{1}{\theta} \left( \frac{f(x+\theta) + f(x-\theta) - 2L}{2} \right).
\]

The rest of the proof follows from the next more general lemma. \( \square \)

**Lemma 7.3** (Riemann Lebesgue Lemma). Let \( f : [a, b] \to \mathbb{R} \) be a Riemann integrable function, and let \( \lambda_n \in \mathbb{R} \) be a sequence of numbers with \( \lambda_n \to \infty \). Then:

\[
\int_{a}^{b} \sin(\lambda_n x) f(x) \, dx \to 0.
\]

**Proof.** Let \( \varphi : [a, b] \to \mathbb{R} \) be a continuously differentiable function which vanishes at \( x = a, b \) and such that \( \int_{a}^{b} |\varphi| \, dx < \epsilon/2 \). Then the triangle inequality and integrating by parts one time gives:

\[
|\int_{a}^{b} \sin(\lambda_n x) f(x) \, dx| \leq |\int_{a}^{b} \sin(\lambda_n x) \varphi(x) \, dx| + |\int_{a}^{b} \sin(\lambda_n x) (f(x) - \varphi(x)) \, dx|,
\]

\[
\leq \frac{1}{\lambda_n} \int_{a}^{b} \cos(\lambda_n x) \varphi'(x) \, dx + \int_{a}^{b} |f(x) - \varphi(x)| \, dx,
\]

\[
\leq \frac{1}{\lambda_n} \int_{a}^{b} |\varphi'(x)| \, dx + \epsilon/2.
\]

Choosing \( N \) so large that \( n > N \) implies \( \frac{1}{\lambda_n} \int_{a}^{b} |\varphi'(x)| \, dx < \epsilon/2 \) gives the result. \( \square \)

We do not state a theorem which complements the one on uniform convergence if Fourier series.

**Theorem 7.4.** Let \( f : \mathbb{R} \to \mathbb{R} \) be \( 2\pi \)-periodic and piecewise continuously differentiable in the sense that there exists a partition \( P = \{0 = t_0 < \ldots < t_n = 2\pi\} \) with \( f'(x) \) exists and uniformly continuous on each \( (t_{k-1}, t_k) \). Then for each \( x \in \mathbb{R} \) one has the identity:

\[
S_N(f)(x) \to \frac{1}{2} (f(x^+) + f(x^-)), \quad \text{where} \quad f(x^\pm) = \lim_{y \to x^\pm} f(y).
\]
In other words for such $f$ one has $S_N(f)(x) \to f(x)$ at each point of continuity, while $S_N(f)(x)$ converges to the average value at each jump discontinuity.

**Proof.** By the previous theorem it suffices to show that each of $h_{\pm}(\theta) = \frac{f(x_{\pm}) - f(x_{\mp})}{\theta}$ is Riemann integrable on $[0, \pi]$. To this end it suffices to show that $h_{\pm}(\theta)$ is uniformly bounded on $[0, \pi]$, because it is continuous and hence integrable on every interval of the form $[\epsilon, \pi]$ for $\epsilon > 0$.

If $x$ is a point of differentiability of $f$, then the boundedness of $h_{\pm}$ is clear by the mean value theorem (just as in previous proofs). On the other hand suppose $x$ is a jump point for $f$. Let $\theta > 0$ be such that $x + \theta$ lies within the interval if differentiability for $f$ with left endpoint $x$, and let $y$ be in this interval with $x < y < x + \theta$. Then by the mean value theorem we have:

$$f(x + \theta) - f(y) = f'(\xi(y; \theta))(x + \theta - y), \quad \text{where} \quad y < \xi(y; \theta) < x + \theta.$$

Taking an appropriate limit (possibly along a subsequence) of $y \to x^+$, and using the uniform continuity of $f'$, there exists some bounded function $q_+(\theta)$ such that:

$$f(x + \theta) - f(x^+) = q_+(\theta) \theta.$$

Similarly there is a bounded function $q_-$ defined on the interval differentiability of $f$ with right endpoint $x$ such that:

$$f(x - \theta) - f(x^-) = q_-(\theta) \theta.$$

Combining these two facts shows $h_{\pm}(\theta)$ defined above is integrable on $[0, \pi]$. \hfill \Box

### 7.1. The Gibbs Phenomenon.

We now wish to investigate a bit further exactly what happens close to a jump discontinuity. We know that the convergence of $S_N(f)$ to $f(x)$ (assuming $f$ is defined as its average at these points) cannot be uniform because each $S_N(f)$ is continuous at $x$ while $f$ is not. However, what is perhaps a bit surprising is that if one normalizes away the jump by subtracting off the gap value from both $S_N(f)$ and $f$, then the convergence of $S_N(f)$ to $f$ is still not uniform! In fact the loss of uniformity can be measured in a very precise way which leads to the celebrated “Gibbs tails” of $S_N(f)$.

To see what these look like we choose a sequence of points $x_N = x + \frac{\lambda}{N}$ for some fixed $\lambda \in \mathbb{R}$. First suppose $\lambda > 0$ and compute:

$$S_N(f)(x_N) = \int_{-\pi}^{\pi} D_N(\theta)f(x_N + \theta)d\theta,$$

$$= \int_{-\frac{\pi}{N}}^{\frac{\pi}{N}} D_N(\theta)(f(x^+) + r_+(\theta + \lambda/N))d\theta + \int_{-\frac{\pi}{N}}^{\frac{\pi}{N}} D_N(\theta)(f(x^-) + r_-(\theta + \lambda/N))d\theta,$$

$$= f(x^+) - (f(x^+) - f(x^-)) \int_{-\frac{\pi}{N}}^{\frac{\pi}{N}} D_N(\theta)d\theta + \int_{-\frac{\pi}{N}}^{\frac{\pi}{N}} D_N(\theta)r_+(\theta + \lambda/N)d\theta + \int_{-\frac{\pi}{N}}^{\frac{\pi}{N}} D_N(\theta)r_-(\theta + \lambda/N)d\theta,$$

where the functions $r_\pm$ are defined just as in the previous proof as:

$$r_\pm(y) = f(y) - f(x^\pm), \quad \text{where} \quad \pm(y - x) > 0.$$

In particular:

$$r_+(\theta + \lambda/N) = \lambda/N q_+(\theta + \lambda/N) + \theta q_+(\theta + \lambda/N), \quad \text{when} \quad \theta > -\lambda/N,$$

where $q_+$ is a bounded integrable function. A similar identity holds for $r_-(\theta + \lambda/N)$. Therefore using methods similar to those of the previous theorems we have both:

$$\int_{-\frac{\pi}{N}}^{\frac{\pi}{N}} D_N(\theta)r_+(\theta + \lambda/N)d\theta \to 0,$$

$$\int_{-\frac{\pi}{N}}^{\frac{\pi}{N}} D_N(\theta)r_-(\theta + \lambda/N)d\theta \to 0.$$
Writing things out and using a L’Hospital’s rule type of calculation we have:

\[
\frac{1}{N} D_N \left( \frac{1}{N} \theta \right) = \frac{\sin \left( (1 + \frac{1}{2N}) \theta \right)}{2\pi N \sin \left( \frac{1}{2N} \theta \right)} \to \frac{1}{\pi} \frac{\sin(\theta)}{\theta} .
\]

With a little more work one can show:

\[
I(\lambda, N) \to \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(\theta)}{\theta} d\theta = \frac{1}{2} - \frac{1}{\pi} \text{Si}(\lambda) , \quad \text{where } \text{Si}(x) = \int_{0}^{x} \frac{\sin(\theta)}{\theta} d\theta .
\]

The function \(\text{Si}(x)\) is a well known special function. It has the properties that \(\text{Si}(0) = 0\) while \(\lim_{x \to \infty} \text{Si}(x) = \frac{\pi}{2}\). Its critical points are where \(x = k\pi\) for \(x \in \mathbb{Z}\), and at \(x = \pi\) it has a global max. At this point \(x = \pi\) we have approximately:

\[
\frac{1}{\pi} \text{Si}(\pi) - \frac{1}{2} \approx 0.089489872236
\]

Thus, combining with the previous calculations we have for \(x_N = x + \frac{x}{N}\) the maximum Gibbs tail of:

\[
S_N(f)(x_N) = f(x^+) + (f(x^+) - f(x^-)) \frac{1}{\pi} \left( \text{Si}(\pi) - \frac{1}{2} \right) + o(1) \approx f(x^+) + (0.089489872236)(f(x^+) - f(x^-)) .
\]

In other words the partial sums \(S_N(f)(x_N)\) overshoot by about 9% of the gap. This phenomena is symmetric.

Using an analogous calculation one also has for \(x_N = x - \frac{x}{N}\):

\[
S_N(f)(x_N) = f(x^-) + (f(x^-) - f(x^+)) \frac{1}{\pi} \left( \text{Si}(\pi) - \frac{1}{2} \right) + o(1) .
\]

Note that these fluctuations disappear once one takes the average.

Finally it’s worth pointing out that these Gibbs tails are consistent with mean square convergence. Just a little more work shows that the Gibbs tails for jump discontinuities are localized to an \(O(1/N)\) neighborhood of the jump in the sense that if \(x_N \to x\) is any sequence with \(N|x_N - x| \to \infty\) then \(|S_N(f)(x_N) - f(x_N)| \to 0\). In particular the portion where \(|S_N(f) - f|^2 \to 0\) is not uniform only contributes an \(O(1/N)\) amount upon integration, and this is sharp.