FACT SHEET FOR 20E FINAL

Remember to also bring a copy of the first fact sheet because we will not repeat those formulas here.

1. FROM CHAPTER 3

- The second order Taylor expansion for a scalar function $f(x_1, \ldots, x_n)$ of $n$ variables at the point $(x_1^0, \ldots, x_n^0) \in \mathbb{R}^n$ is:

$$f(x_1, \ldots, x_n) \approx f(x_1^0, \ldots, x_n^0) + \sum_{i=1}^{n} \partial_{x_i} f(x_1^0, \ldots, x_n^0)(x_i - x_i^0) + \frac{1}{2} \sum_{i,j=1}^{n} \partial_{x_i} \partial_{x_j} f(x_1^0, \ldots, x_n^0)(x_i - x_i^0)(x_j - x_j^0).$$

In general the error here is on the order of $\| (x_1, \ldots, x_n) - (x_1^0, \ldots, x_n^0) \|^3$, i.e. the error is at most of cubic order in terms of the distance between $(x_1, \ldots, x_n)$ and $(x_1^0, \ldots, x_n^0)$.

2. FROM CHAPTER 7

- A surface integral is:

$$\int \int_S g \, dA ,$$

where $S$ is some portion of a surface in 3D, $g = g(x, y, z)$ is a scalar function, and $dA$ is the surface area element.

There are two concrete cases where we have computed $dA$. The first is where $S$ is given as the graph of a function $z = f(x, y)$ over a domain $D \subseteq \mathbb{R}^2$. Then the surface integral becomes:

$$\int \int_S g \, dA = \int \int_D g(x, y, f(x, y)) \sqrt{1 + f_x^2 + f_y^2} \, dx dy .$$

The other case is when $S$ is parameterized by the vector valued function $(x, y, z) = \Phi(u, v)$, where $(u, v) \in D \subseteq \mathbb{R}^2$. Then the integral becomes:

$$\int \int_S g \, dA = \int \int_D g(\Phi(u, v)) \| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \| \, du dv .$$

- A flux integral is:

$$\int \int_S \vec{F} \cdot \hat{n} \, dA ,$$

where $S$ is some portion of a surface in 3D, $\hat{n}$ is a unit normal along $S$, $\vec{F}$ is a vector field, and $dA$ is the surface area element. Note that the direction of $\hat{n}$ determines the sign of this integral.

There are two concrete cases where we have computed this. The first is where $S$ is given as the graph of a function $z = f(x, y)$ over $D \subseteq \mathbb{R}^2$. Then the outward flux integral (i.e. $\hat{n}$ has a positive $z$ component) becomes:

$$\int \int_S \vec{F} \cdot \hat{n} \, dA = \int \int_D \vec{F}(x, y, f(x, y)) \cdot (-f_x, -f_y, 1) \, dx dy .$$
The other is when $\mathcal{S}$ is parameterized by the function $(x, y, z) = \vec{\Phi}(u, v)$, for $(u, v) \in D \subseteq \mathbb{R}^2$. Then the flux integral becomes:

$$\iint_{\mathcal{S}} \vec{F} \cdot \hat{n} \, dA = \iint_{D} \vec{F}(\vec{\Phi}(u, v)) \cdot (\vec{\Phi}_u \times \vec{\Phi}_v) \, dudv \, .$$

In this case one needs to pay attention to the direction of the normal vector $\hat{n} = \frac{\vec{\Phi}_u \times \vec{\Phi}_v}{\| \vec{\Phi}_u \times \vec{\Phi}_v \|}$.

3. From Chapter 4

- The divergence of a vector field is:

$$\nabla \cdot \vec{F} = \partial_x F_1 + \partial_y F_2 + \ldots + \partial_x F_n \, .$$

Here $F_i$ are the components of $\vec{F}$. The divergence measures how much the flow curves of $\vec{F}$ are expanding or contracting a small cube of fluid.

- The curl of a 3D vector field is:

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} = (\partial_y F_3 - \partial_z F_2, \partial_z F_1 - \partial_x F_3, \partial_x F_2 - \partial_y F_1) \, .$$

The quantity $\vec{\omega} \cdot (\nabla \times \vec{F})$ measures how quickly a fluid with velocity $\vec{F}$ will rotate a small paddle wheel perpendicular to the $\vec{\omega}$ (unit vector) direction, where the sign depends on the “right hand rule”. Since the measurement is taken with the ratio of length to area, the quantity $\vec{\omega} \cdot (\nabla \times \vec{F})$ is actually twice the angular velocity of this wheel as it rotates with the fluid.

- The curl of a vector field $\vec{F}$ also indicates if $\vec{F} = \nabla f$ for some scalar function $f$. One has $\nabla \times \vec{F} = \vec{0}$ iff such an $f$ exists.

4. From Chapter 8

- The Green’s Theorem in 2D says that:

$$\int_{\partial D} \vec{F} \cdot ds = \iint_{D} (\partial_x F_2 - \partial_y F_1) \, dA \, , \quad \text{where} \quad \vec{F} = (F_1, F_2) \, ,$$

and where $\partial D$ is the closed curve that bounds the domain $D \subseteq \mathbb{R}^2$. Here the orientation of $\partial D$ in the path integral is counter-clockwise.

- The Stokes’ Theorem in 3D says that:

$$\int_{\partial S} \vec{F} \cdot ds = \iint_{S} (\nabla \times \vec{F}) \cdot \hat{n} \, dA \, ,$$

where $\partial S$ is the boundary curve of the surface $S \subseteq \mathbb{R}^3$, and the orientation of $\partial S$ in the path integral as well as the direction of $\hat{n}$ obeys the “right-hand-rule”. This should be understood as a generalization of the Green’s theorem.

Notice that Stokes’ Theorem also says:

$$\iint_{\mathcal{S}_1} (\nabla \times \vec{F}) \cdot \hat{n} \, dA = \iint_{\mathcal{S}_2} (\nabla \times \vec{F}) \cdot \hat{n} \, dA \, ,$$

as long as $\mathcal{S}_1$ and $\mathcal{S}_2$ are any two surfaces with the same boundary curve and $\hat{n}$ has a similar orientation in both cases.
• The Divergence Theorem in 3D says that:

\[ \int_S \mathbf{F} \cdot \hat{n} \, dA = \int_R \nabla \cdot \mathbf{F} \, dV , \]

where \( S = \partial R \) is the closed surface that bounds the region \( R \subseteq \mathbb{R}^3 \). Here \( \hat{n} \) points to the exterior of \( R \).