FACT SHEET FOR 20E FINAL

Here are the combined fact sheets along with a few minor corrections.

1. FROM CHAPTER 2

• The derivative matrix of the vector-valued function \( \Phi: \mathbb{R}^n \to \mathbb{R}^k \) is the \( k \times n \) matrix:
  \[
  D\Phi|_{x=x_0} = (a_{ij}) = (\partial_j f_i)|_{x=x_0},
  \]
  where \( x_0 \in \mathbb{R}^n \) and the \( f_i \) are the component functions of \( \Phi = (f_1, \ldots, f_k) \).

• The general chain rule for the composition of two vector valued functions is (just matrix multiplication):
  \[
  D(\Psi \circ \Phi)|_{x=x_0} = D\Psi|_{\Phi(x_0)} \cdot D\Phi|_{x=x_0}.
  \]

• The linearization of a vector valued function \( \Phi \) at \( x_0 \) is:
  \[
  \Phi(x) \approx \Phi(x_0) + D\Phi|_{x_0} \cdot (x - x_0).
  \]
  In general the error here is on the order of \( \| x - x_0 \|_2 \). Note also that the right hand side of the above formula is exactly the expression for the tangent plane of \( \Phi \) at \( x_0 \).

2. FROM CHAPTER 3

• The second order Taylor expansion for a scalar function \( f(x_1, \ldots, x_n) \) of \( n \) variables at the point \( (x_1^0, \ldots, x_n^0) \in \mathbb{R}^n \) is:
  \[
  f(x_1, \ldots, x_n) \approx f(x_1^0, \ldots, x_n^0) + \sum_{i=1}^{n} \partial_i f(x_1^0, \ldots, x_n^0)(x_i - x_i^0) + \frac{1}{2} \sum_{i,j=1}^{n} \partial_i \partial_j f(x_1^0, \ldots, x_n^0)(x_i - x_i^0)(x_j - x_j^0).
  \]
  In general the error here is on the order of \( \| (x_1, \ldots, x_n) - (x_1^0, \ldots, x_n^0) \|_3 \), i.e. the error is at most of cubic order in terms of the distance between \( (x_1, \ldots, x_n) \) and \( (x_1^0, \ldots, x_n^0) \).

3. FROM CHAPTER 4

• The divergence of a vector field is:
  \[
  \nabla \cdot \vec{F} = \partial_{x_1} F_1 + \partial_{x_2} F_2 + \ldots + \partial_{x_n} F_n.
  \]
  Here \( F_i \) are the components of \( \vec{F} \). The divergence measures how much the flow curves of \( \vec{F} \) are expanding or contracting a small cube of fluid.

• The curl of a 3D vector field is:
  \[
  \nabla \times \vec{F} = \begin{vmatrix}
  \partial_x & \partial_y & \partial_z \\
  F_1 & F_2 & F_3
  \end{vmatrix} = (\partial_y F_3 - \partial_z F_2, \partial_z F_1 - \partial_x F_3, \partial_x F_2 - \partial_y F_1).
  \]
  The quantity \( \vec{\omega} \cdot (\nabla \times \vec{F}) \) measures how quickly a fluid with velocity \( \vec{F} \) will rotate a small paddle wheel perpendicular to the \( \vec{\omega} \) (unit vector) direction, where the sign depends on the “right hand rule”. Since the measurement is taken with the ratio of length to area, the quantity \( \vec{\omega} \cdot (\nabla \times \vec{F}) \) is actually \( \text{twice} \) the angular velocity of this wheel as it rotates with the fluid.

• The curl of a vector field \( \vec{F} \) also indicates if \( \vec{F} = \nabla f \) for some scalar function \( f \). One has \( \nabla \times \vec{F} = \vec{0} \) iff such an \( f \) exists.
4. From Chapter 6

- If \((x, y) = \Phi(u, v)\) is a one to one and onto map between two domains \(D^*\) and \(D = \Phi(D^*)\) in \(\mathbb{R}^2\), then the change of variables formula for integrals is:

\[
\int_D^* f(\Phi(u, v)) \left| \frac{\partial\Phi}{\partial(u, v)} \right| \, du \, dv = \int_D f(x, y) \, dx \, dy .
\]

Here \(\left| \frac{\partial\Phi}{\partial(u, v)} \right|\) is the absolute value of the determinant of the matrix of first partial derivatives:

\[
\left| \frac{\partial\Phi}{\partial(u, v)} \right| = \left| \partial_u x\partial_v y - \partial_u y\partial_v x \right| .
\]

We are writing \(\Phi(u, v) = (x(u, v), y(u, v))\).

- There are two very important particular cases of the above formula. The first is polar coordinates:

\[
x = r \cos(\theta) , \quad y = r \sin(\theta) ,
\]

where \(\frac{\partial(x, y)}{\partial(r, \theta)} = r \, dr \, d\theta\). For example:

\[
\int_{x^2 + y^2 \leq R^2} f(x, y) \, dx \, dy = \int_0^{2\pi} \int_0^R f(r \cos(\theta), r \sin(\theta)) \, r \, dr \, d\theta .
\]

- The other main important case is the analog of the above formula in three dimensions:

\[
x = \rho \sin(\phi) \cos(\theta) , \quad y = \rho \sin(\phi) \sin(\theta) , \quad z = \rho \cos(\phi) ,
\]

where \(\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta\). For example:

\[
\int_{x^2 + y^2 + z^2 \leq R^2} f(x, y, z) \, dx \, dy \, dz =
\int_0^{2\pi} \int_0^\pi \int_0^R f(\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi)) \, \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta .
\]

5. From Chapter 7

- A path integral is:

\[
\int_{c} f ds = \int_{a}^{b} f(x(t), y(t)) \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} \, dt , \quad \text{where } \vec{c}(t) = (x(t), y(t)) ,
\]

in 2D, and:

\[
\int_{c} f ds = \int_{a}^{b} f(x(t), y(t), z(t)) \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)} \, dt , \quad \text{where } \vec{c}(t) = (x(t), y(t), z(t)) ,
\]

in 3D. Here \(f\) is a scalar function, and \(\vec{c}(t)\) is a curve in the plane or space.

- A line integral is:

\[
\int_{c} \vec{F} \cdot ds = \int_{a}^{b} F(\vec{c}(t)) \cdot \vec{c}'(t) \, dt ,
\]

where \(\vec{F}\) is a vector field 2D or 3D and \(\vec{c}(t)\) is a curve in 2D or 3D (same dimension for both).
• A **parametrized surface** in 3D is given by a one to one and onto vector valued function \( \Phi : D \to \mathbb{R}^3 \) where \( D \subseteq \mathbb{R}^2 \). Here \( \Phi(u, v) = (x(u, v), y(u, v), z(u, v)) \). The vectors:

\[
\vec{T}_u = \frac{\partial \Phi}{\partial u}, \quad \vec{T}_v = \frac{\partial \Phi}{\partial v}, \quad \vec{n} = \vec{T}_u \times \vec{T}_v,
\]

are (in order) the tangent vectors to the surface in the directions of increasing \( u \) and \( v \), and the
normal vector to the surface. Here \( \vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \) is the vector cross product. We assume that \( \vec{n} \neq \vec{0} \), which is the definition of a **regular surface**.

One can use \( \vec{n} \) to write the tangent plane to the surface at \( (x_0, y_0, z_0) = \Phi(u_0, v_0) \) via:

\[
\vec{n} \cdot (x - x_0, y - y_0, z - z_0) = 0,
\]

assuming the surface is regular at \( (x_0, y_0, z_0) \).

• A **surface area** of a parametrized surface \( \Phi : D \to \mathbb{R}^3 \) is given by:

\[
\text{Area} = \iint_D \| \vec{\Phi}_u \times \vec{\Phi}_v \| \, dudv.
\]

In the special case where the surface is the graph of a function \( z = f(x, y) \) this formula becomes:

\[
\iint_D \sqrt{1 + f_x^2 + f_y^2} \, dxdy.
\]

• A **surface integral** is:

\[
\iint_S g \, dA,
\]

where \( S \) is some portion of a surface in 3D, \( g = g(x, y, z) \) is a scalar function, and \( dA \) is the surface area element.

There are two concrete cases where we have computed \( dA \). The first is where \( S \) is given as the graph of a function \( z = f(x, y) \) over a domain \( D \subseteq \mathbb{R}^2 \). Then the surface integral becomes:

\[
\iint_S g \, dA = \iint_D g(x, y, f(x, y)) \sqrt{1 + f_x^2 + f_y^2} \, dxdy.
\]

The other case is when \( S \) is parameterized by the vector valued function \( (x, y, z) = \Phi(u, v) \), where \( (u, v) \in D \subseteq \mathbb{R}^2 \). Then the integral becomes:

\[
\iint_S g \, dA = \iint_D g(\Phi(u, v)) \| \vec{\Phi}_u \times \vec{\Phi}_v \| \, dudv.
\]

• A **flux integral** is:

\[
\iint_S \vec{F} \cdot \vec{n} \, dA,
\]

where \( S \) is some portion of a surface in 3D, \( \vec{n} \) is a unit normal along \( S \), \( \vec{F} \) is a vector field, and \( dA \) is the surface area element. Note that the direction of \( \vec{n} \) determines the sign of this integral.

There are two concrete cases where we have computed this. The first is where \( S \) is given as the graph of a function \( z = f(x, y) \) over \( D \subseteq \mathbb{R}^2 \). Then the **outward flux integral** (i.e. \( \vec{n} \) has a positive \( z \) component) becomes:

\[
\iint_S \vec{F} \cdot \vec{n} \, dA = \iint_D \vec{F}(x, y, f(x, y)) \cdot (-f_x, -f_y, 1) \, dxdy.
\]
The other is when $S$ is parameterized by the function $(x, y, z) = \Phi(u, v)$, for $(u, v) \in D \subseteq \mathbb{R}^2$. Then the flux integral becomes:

$$\iint_S \vec{F} \cdot \hat{n} \ dA = \iint_D \vec{F}(\Phi(u, v)) \cdot (\Phi_u \times \Phi_v) \ dudv .$$

In this case one needs to pay attention to the direction of the normal vector $\hat{n} = \frac{\Phi_u \times \Phi_v}{\| \Phi_u \times \Phi_v \|}$.

6. From Chapter 8

- The Green’s Theorem in 2D says that:

$$\int_{\partial D} \vec{F} \cdot ds = \iint_D \left( \partial_x F_2 - \partial_y F_1 \right) \ dA , \quad \text{where} \quad \vec{F} = (F_1, F_2) ,$$

and where $\partial D$ is the closed curve that bounds the domain $D \subseteq \mathbb{R}^2$. Here the orientation of $\partial D$ in the path integral is counter-clockwise.

- The Stokes’ Theorem in 3D says that:

$$\int_{\partial S} \vec{F} \cdot ds = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \ dA ,$$

where $\partial S$ is the boundary curve of the surface $S \subseteq \mathbb{R}^3$, and the orientation of $\partial S$ in the path integral as well as the direction of $\hat{n}$ obeys the “right-hand-rule”. This should be understood as a generalization of the Green’s theorem.

Notice that Stokes’ Theorem also says:

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot \hat{n} \ dA = \iint_{S_2} (\nabla \times \vec{F}) \cdot \hat{n} \ dA ,$$

as long as $S_1$ and $S_2$ are any two surfaces with the same boundary curve and $\hat{n}$ has a similar orientation in both cases.

- The Divergence Theorem in 3D says that:

$$\iint_S \vec{F} \cdot \hat{n} \ dA = \iiint_R \nabla \cdot \vec{F} \ dV ,$$

where $S = \partial R$ is the closed surface that bounds the region $R \subseteq \mathbb{R}^3$. Here $\hat{n}$ points to the exterior of $R$. 

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