MATH 231A HOMEWORK 3

Abstract. Here are some additional problems concerning the wave and Schrödinger equation.

1. Weak Plane Wave Solutions to the Wave Equation

Let \( f \in L^1_{\text{loc}}(\mathbb{R}) \). Show that for any fixed \( \xi_0 \in \mathbb{R}^n \) the function \( u(t,x) = f(\mid \xi_0 \mid t - \xi_0 \cdot x) \) is a weak solution (in the sense of distributions) to the wave equation \( u_{tt} = \Delta u \) on \( \mathbb{R}^{1+n} \).

2. Dispersive Estimate for the Wave Equation

Recall that Kirchhoff’s formulas for the solution to the wave equation \( u_{tt} = \Delta u \) on \( \mathbb{R}^{1+3} \) is given by:

\[
  u(t,x) = \frac{1}{4\pi t^2} \int_{|x-y|=t} (y-x) \cdot \nabla u_0 + u_0(y)dy + \frac{1}{4\pi t} \int_{|x-y|=t} u_1(y)dy,
\]

where \( u_0(x) = u(0,x) \) and \( u_1(x) = \partial_t u(0,x) \). Use the divergence theorem to show that:

\[
  |u(t,x)| \leq C t^{-1} \left( \sum_{j=0}^{2} \| |y-x|^{-j} \partial_x^{2-j} u_0 \|_{L^1(dy)} + \sum_{j=0}^{1} \| |y-x|^{-j} \partial_x^{1-j} u_1 \|_{L^1(dy)} \right),
\]

where \( \partial_x^k \) is shorthand for all partial derivatives \( \partial_x^\alpha \) of order \( |\alpha| = k \).

3. Schrödinger Evolution in \( \mathcal{S}(\mathbb{R}^n) \)

Recall that the Schwartz space \( \mathcal{S}(\mathbb{R}^n) \) is the set of all \( C^\infty \) functions on \( \mathbb{R}^n \) such that \( \| x^\alpha \partial_x^\beta f \|_{L^\infty} < \infty \) for all \( \alpha, \beta \in \mathbb{N}^n \). Recall that the Fourier transform is defined by \( \hat{f}(\xi) = \int e^{-ix \cdot \xi} f(x)dx \).

a) Show that if \( f \in \mathcal{S}(\mathbb{R}^n) \), then for each \( \alpha, \beta \in \mathbb{N}^n \) one has:

\[
  \| \xi^\alpha \partial_x^\beta \hat{f} \|_{L^\infty} \leq C_{\alpha, \beta} \sum_{|\alpha'| \leq |\beta| + n+1} \| x^{\alpha'} \partial_x^{\beta'} f \|_{L^\infty},
\]

where \( C_{\alpha, \beta} \) are universal constants depending only on \( \alpha, \beta \). In particular \( f \mapsto \hat{f} \) is a continuous linear map on \( \mathcal{S}(\mathbb{R}^n) \).

b) Let \( u_0 \in \mathcal{S}(\mathbb{R}^n) \). Show that the function:

\[
  u(t,x) = \frac{1}{(2\pi)^n} \int e^{ix \cdot \xi - t|\xi|^2} \hat{u}_0(\xi)d\xi,
\]

is such that \( u \in C^\infty_c(\mathcal{S}(\mathbb{R}^n)) \), and \( iu_t = -\Delta u \).

c) Show that for the solution (1) one has \( u(t) \to u_0 \) in \( \mathcal{S}(\mathbb{R}^n) \) in the sense that \( \| x^\alpha \partial_x^\beta (u(t) - u_0) \|_{L^\infty} \to 0 \) for each \( \alpha, \beta \in \mathbb{N}^n \).

4. Symmetries of the Schrödinger Equation

Let \( u(t,x) \) be a \( C^2(I;\mathbb{R}^{1+n}) \) solution of \( iu_t = -\Delta u \). Show that the following list of functions \( w \) built from \( u \) also solve the Schrödinger equation:

a) \( w(t,x) = u(t-t_0, x-x_0) \) for any \( (t_0, x_0) \in \mathbb{R}^{1+n} \).

b) \( w(t,x) = u(t, Ox) \) for any \( O \in O(n) \) (orthogonal matrices).

c) \( w(t,x) = u(\lambda^2 t, \lambda x) \) for any real \( \lambda > 0 \).

d) \( w(t,x) = e^{i(x \cdot \xi - t|\xi|^2)} u(t, x - 2\xi_0 t) \) for any \( \xi_0 \in \mathbb{R}^n \).

e) \( w(t,x) = \frac{1}{t^2} e^{i|\xi|^2/2t^2} u(\frac{1}{t}, \xi) \) for \( t > 0 \).
5. GAUSSIAN BEAM SOLUTIONS TO SCHRODINGER EQUATION

Show that the family of functions:

\[ u(t, x) = (1 + 2i\lambda t)^{-\frac{3}{2}} e^{i(x-\xi_0 \cdot t)[\xi_0]^2} e^{-\frac{\lambda}{2\pi} |x-2\xi_0 t|^2}, \quad \lambda > 0, \quad \xi_0 \in \mathbb{R}^n, \]

all solve the Schrodinger equation \( i\partial_t u = -\Delta u. \)