NOTES ON 2ND ORDER ELLIPTIC PDE IN $H^k$ SPACES

JACOB STERBENZ

Abstract. Here are some notes to supplement the material in Evans.

1. Solvability

We start with an abstract solvability theorem.

**Theorem 1.1** (Abstract Solvability). Suppose that $X$ and $Y$ are Banach spaces with $X$ reflexive. Let $Z \subset Y^*$ be a linear subspace. Let $L : X \to Y$ be a bounded linear transformation. Then a sufficient condition that given any $f \in Y$ one can always find $u \in X$ such that:

$$(1) \quad \langle Lu - f, \varphi \rangle = 0, \quad \text{for all } \varphi \in Z,$$

is that there exists an a-priori estimate:

$$(2) \quad \| \varphi \|_{Y^*} \leq C \| L^* \varphi \|_{X^*}, \quad \text{for all } \varphi \in Z.$$ 

**Proof.** The bound (2) implies that $L^*|_Z$ is injective. Given $f \in Y$, on $\text{Range}(L^*|_Z)$ we define the linear functional:

$$\ell(w) = \langle f, \varphi \rangle, \quad \text{when } w = L^* \varphi.$$ 

This is in fact a bounded linear functional on $\text{Range}(L^*|_Z) \subseteq X^*$ thanks to (2) which tells us:

$$|\ell(w)| \leq \| f \|_Y \| \varphi \|_{Y^*} \leq C \| f \|_Y \| L^* \varphi \|_{X^*} = C_f \| w \|_{X^*}.$$ 

By the Hahn-Banach theorem there exists a bounded linear extension $\bar{\ell}$ of $\ell$ to all of $X^*$. Since we are assuming $X^{**} = X$, there must exists some $u \in X$ such that:

$$\bar{\ell}(w) = \langle u, w \rangle, \quad \text{for all } w \in X^*.$$ 

In particular this equation holds when $w = L^* \varphi$ for some $\varphi \in Z$, so varying $\varphi$ we have:

$$\langle u, L^* \varphi \rangle = \langle f, \varphi \rangle, \quad \text{for all } \varphi \in Z.$$ 

This is equivalent to (1). $\square$

The above setup has immediate implications for the solution to the Dirichlet problem.

**Theorem 1.2** (Solution of a Dirichlet Problem). Let $\Omega \subset \subset \mathbb{R}^n$ with $\partial \Omega$ a $C^1$ boundary. Suppose that $a^{ij}, b^i, c \in L^\infty(\Omega)$, and suppose that $a^{ij}$ is uniformly elliptic in the sense that $a^{ij}(x)\xi_i \xi_j \geq c|\xi|^2$ for some $c > 0$ for a.e. $x \in \Omega$. Then there exists a sufficiently large $\lambda_0 = \lambda_0(a,b,c)$ such that for all $\lambda \geq \lambda_0$ the equation:

$$L_{\lambda}u = f, \quad \text{where } u \in H^1_0(\Omega), \quad \text{and } L_{\lambda} = \partial_i a^{ij} \partial_j + b^i \partial_i + c - \lambda$$

always has a (unique) solution for $f \in H^{-1}(\Omega)$. In the case $b^i = c = 0$ we can choose $\lambda = 0$.

**Proof.** Let $X = H^1_0(\Omega)$ and $Y = H^{-1}(\Omega)$. Then $L_{\lambda} : X \to Y$ is continuous and we have that $X$ is reflexive. Let $Z = C^\infty_0(\Omega)$. Then we can apply the previous theorem as long as we can show:

$$(3) \quad \| \varphi \|_{H^1_0(\Omega)} \leq C \| L_{\lambda}^* \varphi \|_{H^{-1}(\Omega)}, \quad \text{for all } \varphi \in C^\infty_0(\Omega).$$
Using the uniform ellipticity of \(a^{ij}\) and integration by parts we have:
\[
\lambda \| \varphi \|_{L^2(\Omega)}^2 + c \| \nabla \varphi \|_{L^2(\Omega)}^2 \leq \lambda \| \varphi \|_{L^2(\Omega)}^2 + \int_{\Omega} a^{ij} \partial_i \varphi \partial_j \varphi dx ,
\]
\[
= - \langle L^* \varphi, \varphi \rangle - \int_{\Omega} \varphi b^i \partial_i \varphi dx + \int_{\Omega} c \varphi^2 dx ,
\]
\[
\leq \| L^* \varphi \|_{H^{-1}(\Omega)} \| \varphi \|_{H^1(\Omega)} + C \| \varphi \|_{H^1(\Omega)} \| \varphi \|_{L^2(\Omega)} .
\]
Here \(c, C\) are fixed but \(\lambda\) is at our disposal. Using “Young’s inequality” \(AB \leq \epsilon^{-1} A^2 + \epsilon B^2\) which holds uniformly in \(\epsilon > 0\) it is clear we can choose \(\lambda\) sufficiently large in the previous estimate so as to achieve (3).

Note that if \(b^i = c = 0\), then we use Poincare’s estimate \(\| \varphi \|_{L^2(\Omega)} \leq C_\Omega \| \nabla \varphi \|_{L^2(\Omega)}\) instead and choose \(\lambda = 0\) in the previous proof.

\[\square\]

2. Interior Regularity

We now know that we can construct weak solutions to the equation \(Lu = f\), but only that \(u \in H_0^1(\Omega)\) even if \(a^{ij}, b^i, c\) and \(f\) are very smooth. To get more control of \(u\) we need to prove some a-priori Sobolev bounds for elliptic equations.

**Lemma 2.1.** Let \(a^{ij} \in C^1(\mathbb{R}^n)\) be symmetric. Let \(\varphi \in C_0^\infty(\mathbb{R}^n)\) and let \(Q_\epsilon u = \varphi \ast u\) where \(\varphi(x) = \epsilon^{-n} \varphi(\epsilon^{-1}x)\). Then for \(u \in H^1(\mathbb{R}^n)\) there exists a constant \(C = C(a, \varphi)\) such that one has the commutator estimate uniformly in \(\epsilon > 0\):
\[
\| [A, Q_\epsilon] u \|_{L^2(\mathbb{R}^n)} \leq C \| \nabla u \|_{L^2(\mathbb{R}^n)} , \quad \text{where} \quad A = \partial_\epsilon a^{ij} \partial_j .
\]

**Proof.** The commutator is given explicitly by the formula:
\[
[A, Q_\epsilon] u(x) = \epsilon^{-1} \int_{\mathbb{R}^n} (a^{ij}(x) - a^{ij}(y)) (\partial_\epsilon \varphi)_\epsilon(x - y) \partial_j u(y) dy + \int_{\mathbb{R}^n} (\partial_\epsilon a^{ij}(x)) \varphi(x - y) \partial_j u(y) dy ,
\]
\[
= \int_{\mathbb{R}^n} K_\epsilon^j(x, y) \partial_j u(y) dy .
\]
By the mean value theorem applied to \(a^{ij}\) we see \(K_\epsilon^j\) is a smooth kernel satisfying:
\[
|K_\epsilon^j(x, y)| \lesssim \epsilon^{-n} \psi(\epsilon^{-1}(x - y)) , \quad \text{where} \quad \int_{\mathbb{R}} \psi(x) dx \lesssim 1 ,
\]
where the implicit constant depends on the \(C^1(\mathbb{R}^n)\) norm of \(a^{ij}\) and the support of \(\varphi\), but not on \(\epsilon\). This last estimate implies both:
\[
\| K_\epsilon^j \|_{L^\infty_{x} L^1_y} \lesssim 1 , \quad \| K_\epsilon^j \|_{L^\infty_y L^1_x} \lesssim 1 ,
\]
with implicit constant depending only on \(\varphi\) and it first derivatives. Then estimate (4) follows from these bounds and Schur’s test. \[\square\]

**Lemma 2.2.** Let \(a^{ij} \in C^1(\Omega)\) be uniformly elliptic. Then there exists a constant \(C = C(a, \Omega)\) such that for all \(\varphi \in C_0^\infty(\Omega)\) one has the a-priori estimate:
\[
\| \varphi \|_{H^2(\Omega)} \leq C( \| A \varphi \|_{L^2(\Omega)} + \| \varphi \|_{L^2(\Omega)} ) , \quad \text{where} \quad A = \partial_\epsilon a^{ij} \partial_j .
\]

**Proof.** First use ellipticity and integration by parts to write:
\[
c \| \nabla \varphi \|_{L^2(\Omega)}^2 \leq - \int_{\Omega} A \varphi \cdot \varphi dx .
\]
Hölder’s and Young’s inequality then gives:
\[
\| \varphi \|_{H^2(\Omega)} \leq C( \| A \varphi \|_{L^2(\Omega)} + \| \varphi \|_{L^2(\Omega)} ) .
\]
Applying (6) to \(\partial \varphi\) and several integration by parts gives:
\[
c \| \nabla \partial \varphi \|_{L^2(\Omega)} \leq \int_{\Omega} A \varphi \cdot \partial^2 \varphi dx - \int_{\Omega} (\partial a^{ij}) \partial_i \varphi \partial_j \varphi dx .
\]
Young’s inequality and the previous line then gives (5). \[\square\]
Proposition 2.3. Let $a^{ij} \in C^1(\Omega)$ be elliptic. Then for each pair of subdomains $V \subset U \subseteq \Omega$ there exists a constant $C = C(a,V,U)$ such that for $u \in H^1_{loc}(\Omega)$:

$$
\| u \|_{H^2(V)} \leq C \left( \| Au \|_{L^2(U)} + \| u \|_{L^2(U)} \right), \quad \text{where } A = \partial_i a^{ij} \partial_j.
$$

Proof. First let $\eta$ be a cutoff function on an intermediate domain $\tilde{U}$, where $V \subset \tilde{U} \subset U$, with support $\eta$ in $U$. Set $\tilde{u} = \eta u$. Applying the same reasoning as in estimate (7), but this time making use of the $H^1_0$ and $H^{-1}$ duality, we get:

$$
c \| \nabla \tilde{u} \|_{L^2(\Omega)}^2 \leq -\langle A \tilde{u}, \tilde{u} \rangle \leq \epsilon^{-1} \| A \tilde{u} \|_{H^{-1}(\Omega)}^2 + \epsilon \| \tilde{u} \|_{H^1(\Omega)}^2.
$$

Expanding out the derivatives and rearranging terms we have the identity:

$$
A \tilde{u} = \eta Au + \partial_i (a^{ij}(\eta)u) + \partial_j (\partial_i(\eta)a^{ij}u) - \partial_j (a^{ij}\partial_i(\eta))u.
$$

This gives:

$$
\| A \tilde{u} \|_{H^{-1}(\Omega)} \lesssim \| Au \|_{L^2(U)} + \| u \|_{L^2(U)},
$$

for an implicit constant depending on $a^{ij}, U, V$. Thus, combining the previous lines gives:

$$
\| u \|_{H^1(U)} \lesssim \| Au \|_{L^2(U)} + \| u \|_{L^2(U)}.
$$

Now redefine $\eta$ be a cutoff function on $V$ with support $\eta$ in $\tilde{U}$, and again set $\tilde{u} = \eta u$. In light of the last line above it suffices to show:

$$
\| \tilde{u} \|_{H^2(\Omega)} \leq C \left( \| A \tilde{u} \|_{L^2(\Omega)} + \| \tilde{u} \|_{H^1(\Omega)} \right).
$$

Let $u_n \to \tilde{u}$ in $H^1(\Omega)$ be a sequence of mollifications $u_n = \varphi_n * u$ where $\epsilon = 1/n$. Combining estimates (4) and (5) we get the uniform bounds:

$$
\| u_n \|_{H^2(\Omega)} \leq C \left( \| A \tilde{u} \|_{L^2(\Omega)} + \| \tilde{u} \|_{H^1(\Omega)} \right).
$$

By weak-* compactness in $H^2(\Omega)$ there exists a subsequence $u_{n_k}$ with $u_{n_k} \to w$ in the sense of distributions with $w \in H^2(\Omega)$. By weak lower semicontinuity of norms we also know $\| u \|_{H^2(\Omega)} \leq \liminf \| u_{n_k} \|_{H^2(\Omega)}$. On the other hand we must have $u_{n_k} \to \tilde{u}$ in $H^1(\Omega)$, so in fact $\tilde{u} = w$ a.e. as measureable functions. This shows (9) and completes the proof.

The previous estimates are stable with respect to the addition of lower order terms:

Proposition 2.4. Let $a^{ij} \in C^1(\Omega)$ be elliptic, and let $b^i, c \in L^\infty(\Omega)$. Then for each pair of subdomains $V \subset U \subseteq \Omega$ there exists a constant $C = C(a,b,c,V,U)$ such that for $u \in H^1_{loc}(\Omega)$:

$$
\| u \|_{H^2(V)} \leq C \left( \| Lu \|_{L^2(U)} + \| u \|_{L^2(U)} \right), \quad \text{where } L = \partial_i a^{ij} \partial_j + b^i \partial_i + c.
$$

Proof. To reduce to (8) it suffices to show a-priori that:

$$
\| u \|_{H^1(V)} \leq C \left( \| Lu \|_{L^2(U)} + \| u \|_{L^2(U)} \right), \quad \text{where } L = \partial_i a^{ij} \partial_j + b^i \partial_i + c,
$$

with the same assumptions as above. Using the same calculations as in the previous proof, this boils down to showing for $\tilde{u} \in H^1(\Omega)$:

$$
\| \tilde{u} \|_{H^1(\Omega)} \leq C \left( \| L\tilde{u} \|_{H^{-1}(\Omega)} + \| \tilde{u} \|_{L^2(\Omega)} \right), \quad \text{when } \text{supp}(\tilde{u}) \subset \Omega.
$$

This last estimate follows from the same integration by parts calculations we have used a number of times now.

Corollary 2.5. Let $a^{ij} \in C^{k+1}(\Omega)$ be elliptic, and suppose $b^i, c \in C^k(\Omega)$. Then for each pair of subdomains $V \subset U \subseteq \Omega$ there exists a constant $C = C(a,b,c,V,U)$ such that for $u \in H^1_{loc}(\Omega)$:

$$
\| u \|_{H^{k+2}(V)} \leq C \left( \| Lu \|_{H^k(U)} + \| u \|_{L^2(U)} \right), \quad \text{where } A = \partial_i a^{ij} \partial_j + b^i \partial_i + c.
$$
Proof. We have already shown this for \( k = 0 \). It remains to show it for \( k \geq 1 \) assuming it is true for \( k \) replaced by \( k - 1 \). Let \( \bar{U} \) be an intermediate domain \( V \subset\subset \bar{U} \subset\subset U \). By the inductive hypothesis we know \( \partial^\alpha u \in H^1_{\text{loc}}(\Omega) \) for \( |\alpha| = k \). In addition, by the Leibniz rule we see (in the sense of \( \mathcal{D}'(\Omega) \)):

\[
L \partial^\alpha u = \partial^\alpha Lu + P(x, D)u,
\]

where \( P(x, D) \) is a partial differential operator of order \( k + 1 \) with \( L^\infty(\Omega) \) coefficients. Note that here we have used \( a^{ij} \in C^{k+1}(\Omega) \) and \( b^i, c \in C^k(\Omega) \). Applying estimate (\( 10 \)) with respect to the pair \( V \subset\subset \bar{U} \) we have we have:

\[
\sum_{|\alpha| = k} \| \partial^\alpha u \|_{H^2(V)} \leq \| Lu \|_{H^1(\bar{U})} + \| u \|_{H^{k+1}(\bar{U})}.
\]

The proof concludes with the inductive hypothesis with respect to the pair \( \bar{U} \subset\subset U \) which gives the desired bound for \( \| u \|_{H^{k+1}(\bar{U})} \).

\[\square\]

3. Boundary Regularity

With a little more work the results of the previous section can be modified to include boundaries. Here we'll replace mollifications with difference quotients which are easier to handle close to the boundary.

First let \( \mathbb{R}^n_+ \) denote the upper half plane \( x^n > 0 \), and denote the variable \( x = (x', x^n) \).

**Proposition 3.1.** Let \( a^{ij} \in C^1(\mathbb{R}^n_+) \) be uniformly elliptic, and let \( b^i, c \in L^\infty(\mathbb{R}^n_+) \). Then there exists a constant \( C = C(a, b, c) > 0 \) such that for all \( u \in H^1_0(\mathbb{R}^n_+) \):

\[
\| u \|_{H^2(\mathbb{R}^n_+)} \leq C(\| Lu \|_{L^2(\mathbb{R}^n_+)} + \| u \|_{L^2(\mathbb{R}^n_+)}), \quad L = \partial_i a^{ij} \partial_j + b^i \partial_i + c.
\]

Proof. Using integration by parts calculations similar to those in the previous section we have:

\[
\| u \|_{H^1(\mathbb{R}^n_+)} \leq C(\| Lu \|_{H^{-1}(\mathbb{R}^n_+)} + \| u \|_{L^2(\mathbb{R}^n_+)}), \quad \text{for } u \in H^1_0(\mathbb{R}^n).
\]

Therefore is suffices to prove (\( 11 \)) with \( L \) replaced by \( A = \partial_i a^{ij} \partial_j \).

Now let \( \delta_h^i u(x) = h^{-1}(u(x + he_i) - u(x)) \) be the difference quotient in direction of the unit vector \( e_i \). Then from \( [\partial_j, \delta_h^i] = 0 \) and \( \delta_h^i(uv)(x) = \delta_h^i u(x)v(x) + u(x + he_i) \delta_h^i v(x) \) we get:

\[
A \delta_h^i u(x) = \delta_h^i f(x) - \partial_i \delta_h^i a^{ij}(x) \partial_j u(x + he_i), \quad \text{when } Au = f.
\]

Note that when \( k = 1, \ldots, n - 1 \) we still have \( \delta_h^i u \in H^1_0(\mathbb{R}^n_+) \). Therefore estimate (\( 12 \)) gives the following uniform bound in \( h \to 0 \):

\[
\| \delta_h^i u \|_{H^1(\mathbb{R}^n_+)} \leq C(\| \delta_h^i Au \|_{H^{-1}(\mathbb{R}^n_+)} + \| \delta_h^i u \|_{L^2(\mathbb{R}^n_+)} + \| u \|_{H^1(\mathbb{R}^n_+)}), \quad \text{for } k = 1, \ldots, n - 1.
\]

Next, for \( \varphi \in C^\infty_0(\mathbb{R}^n) \) we have \( h^{-\alpha} \delta_h^i \varphi \|_{L^2} \leq \| \partial \varphi \|_{L^2} \) which follows easily from Plancherel's inequality and \( |e^{ih\xi} - 1| \leq |h\xi| \). By density this estimate is also true for all \( u \in H^1_0(\mathbb{R}^n_+) \). As a consequence, if \( v \in H^1_0(\mathbb{R}^n_+) \) we get:

\[
\| \delta_h^i f \|_{L^2(\mathbb{R}^n_+)} = \| (f, \delta_h^i v) \|_{L^2(\mathbb{R}^n_+)} \leq \| f \|_{L^2(\mathbb{R}^n_+)} \| v \|_{H^1(\mathbb{R}^n_+)} = \| \delta_h^i f \|_{H^{-1}(\mathbb{R}^n_+)} \leq \| f \|_{L^2(\mathbb{R}^n_+)} \quad \text{uniformly for } h \to 0.
\]

Therefore, combining with (\( 12 \)) the previous estimate gives:

\[
\| \partial_h^i u \|_{H^1(\mathbb{R}^n_+)} \leq C(\| Au \|_{L^2(\mathbb{R}^n_+)} + \| u \|_{L^2(\mathbb{R}^n_+)}), \quad \text{for } k = 1, \ldots, n - 1.
\]

Now \( \delta_h^i u \to \partial_i u \) weakly in \( H^1(\mathbb{R}^n_+) \) as \( h \to 0 \). By LSC of the \( H^1(\mathbb{R}^n_+) \) norm, this gives:

\[
\| \partial_i u \|_{H^1(\mathbb{R}^n_+)} \leq C(\| Au \|_{L^2(\mathbb{R}^n_+)} + \| u \|_{L^2(\mathbb{R}^n_+)}), \quad \text{for } k = 1, \ldots, n - 1.
\]

Finally, we need to estimate \( \| \partial^2_{ij} u \|_{L^2(\mathbb{R}^n_+)} \). This follows from the fact that ellipticity implies \( a^{nn} \geq c > 0 \) so therefore:

\[
c|\partial^2_{ij} u| \leq |Au| + C(|\nabla u| + \sum_{2 \leq j + k < 2n} |\partial_j \partial_k u|).
\]

Squaring and integrating completes the proof. \( \square \)

**Proposition 3.2.** Let \( \Omega \subset\subset \mathbb{R}^n \) with \( \partial \Omega \) a \( C^2 \) boundary. Let \( a^{ij} \in C^1(\Omega) \) be uniformly elliptic and let \( b^i, c \in L^\infty(\Omega) \). Then there exists a constant \( C = C(a, b, c) \) such that for \( u \in H^1_0(\Omega) \) one has:

\[
\| u \|_{H^2(\Omega)} \leq C(\| Lu \|_{L^2(\Omega)} + \| u \|_{L^2(\Omega)}), \quad \text{where } L = \partial_i a^{ij} \partial_j + b^i \partial_i + c.
\]
Proof. We again have:
\[ \| u \|_{H^1(\Omega)} \leq C(\| Lu \|_{H^{-1}(\Omega)} + \| u \|_{L^2(\Omega)}), \quad \text{for } u \in H_0^1(\Omega). \]
This allows us to trade \( L \) for \( A \) when proving estimate (14), and furthermore to use a partition of unity to reduce matters to an interior estimate and a series of estimates localized close to points where the boundary \( \partial \Omega \) is a \( C^2 \) graph.

The interior portion of estimate (14) was already handled in the previous section. Let \( x_0 \in \partial \Omega \) and let \( \Phi \) be a \( C^2 \) map with \( \Phi(x_0) = 0 \) and sending \( \Omega \) to \( \mathbb{R}^n_+ \) and \( \partial \Omega \) to \( y^n = 0 \) close to \( x_0 \). Changing variables to \( y = \Phi(x) \) we have \( \partial_{x_i} = \frac{\partial \Phi_i}{\partial x} \partial_{y_i} \). Therefore in the \( y \) variables we have:
\[ A = \partial_i \tilde{a}^{ij} \partial_j + \tilde{b}^i \partial_i \]
\[ \tilde{a}^{ij} = a^{ij} \frac{\partial \Phi_i}{\partial x} \frac{\partial \Phi_j}{\partial x}, \quad \tilde{b}^i = (\partial_i \frac{\partial \Phi_i}{\partial x}) a^{ij} \frac{\partial \Phi_j}{\partial x}. \]
The key observation now is that \( \tilde{b}^i \in L^\infty(\mathbb{R}^n_+) \) (after an extension), while:
\[ \tilde{a}^{ij} \xi_i \xi_j \geq c_i (|\frac{\partial \Phi_i}{\partial x}|^2 \geq \overline{c} |\xi|^2), \quad \text{close to } y = 0, \]
where the second inequality follows from the fact that \( \frac{\partial \Phi_i}{\partial x} \) is nonsingular and continuous. Therefore extending \( \tilde{a}^{ij} \) we have the assumptions of Proposition 3.1 and hence estimate (14) after changing back to the \( x \) variables.

Corollary 3.3. Let \( \Omega \subset \subset \mathbb{R}^n \) with \( \partial \Omega \) a \( C^{k+2} \) boundary. Let \( a^{ij} \in C^{k+1}(\overline{\Omega}) \) be elliptic, and assume \( b^i, c \in C^k(\overline{\Omega}) \). Then there exists a constant \( C = C(a, b, c) \) such that for \( u \in H_0^1(\Omega) \):
\[ \| u \|_{H^{k+2}(\Omega)} \leq C(\| L u \|_{H^k(\Omega)} + \| u \|_{L^2(\Omega)}), \quad \text{where } L = \partial_i a^{ij} \partial_j + b^i \partial_i + c. \]

Proof. The proof is by induction and estimate (14). We already know (15) in the interior so it suffices to prove it close to \( \partial \Omega \). By the inductive hypothesis we can also drop the lower order terms and assume \( L = A = \partial_i a^{ij} \partial_j \). Let \( f \) be a local defining function for \( \partial \Omega \) and \( X_1, \ldots, X_{n-1} \) a collection of linearly independent \( C^{k+1} \) vector fields with \( X_i f = 0 \) close to \( \partial \Omega \). Then \( X_i u \in H_0^1(\Omega) \) (by induction), and in addition \( [X_i, A] = P(x, D) \) where \( P \) is a partial differential operator of order 2 with \( C^{k+1}(\overline{\Omega}) \) coefficients. By the inductive hypothesis this gives:
\[ \sum_i \| X_i u \|_{H^{k+1}(\Omega)} \leq C(\| A u \|_{H^k(\Omega)} + \| u \|_{H^{k+1}(\Omega)}), \]
It remains to estimate \( X_n^2 u \) in \( H^k(\Omega) \) where \( X_n = \nabla f \). Using ellipticity we can write:
\[ \sum_{|\alpha| \leq k} \| \partial^\alpha X_n^2 u \| \leq C \left( \sum_{|\alpha| \leq k} \| \partial^\alpha A u \| + \sum_{|\alpha| \leq k+1} (\| \partial^\alpha u \| + \sum_i \| \partial^\alpha X_i u \|) \right). \]
Then (15) follows from integrating the square of this last line and (16).

4. TIY IT ALL TOGETHER

By concatenating Sobolev embeddings we have the following:

Proposition 4.1. Let \( \Omega \subset \subset \mathbb{R}^n \) with \( \partial \Omega \) a \( C^1 \) boundary. Suppose \( u \in H^k(\Omega) \) for \( k > \frac{n}{2} \). Then if \( n \) is odd, after a possible change on a set of measure zero, one has \( u \in C^{l, \frac{1}{2}}(\overline{\Omega}) \) where \( l \) is the greatest integer less than \( k - \frac{n}{2} \). If \( n \) is even then \( u \in C^{k-1, \frac{1}{2}}(\overline{\Omega}) \) for all \( 0 \leq l < 1 \).

Proof. Suppose \( n \) is odd. Then \( k - l = \frac{n-1}{2} + 1 \). By applying the \( W^{1,p} \) Sobolev embeddings \( \frac{n-1}{2} \) times we find that \( H^{\frac{n-1}{2}}(\Omega) \subset L^{2n}(\Omega) \). On the other hand \( W^{1,2n}(\Omega) \subset C^{0, \frac{1}{2}}(\overline{\Omega}) \). Therefore we know that \( \partial^\alpha u \in C^{0, \frac{1}{2}}(\overline{\Omega}) \) for all \( |\alpha| \leq l \), albeit in the sense of distributions. The proof then concludes by induction and the next Lemma.

In the case where \( n \) is even we instead use that \( H^{\frac{n-1}{2}}(\Omega) \subset L^n(\Omega) \), followed by \( W^{2,n}(\Omega) \subset C^{0, \frac{1}{2}}(\overline{\Omega}) \) for all \( 0 \leq l < 1 \) and

There is a subtle issue in the previous proof, namely to show that if \( u \in D'(\Omega) \) and \( \partial_i u \in C(\overline{\Omega}) \) in the sense of distributions, then \( u \) is in fact classically differentiable at every point.
Lemma 4.2. Suppose \( u \in \mathcal{D}'(\Omega) \) and \( \partial_t u \in C(\overline{\Omega}) \) in the sense of distributions. Then (possibly after a change on a set of measure zero) one has that \( \lim_{h \to 0} \delta^h_i u(x) \) exists for all \( x \in \Omega \) and equals \( \partial_i u(x) \), where the latter is taken in the sense of distributions.

Proof. Let \( w_i = \partial_i u \) in the sense of distributions. Then \( \partial_i w_j = \partial_j w_i \) in the sense of distributions. This identity also holds for mollifications, so by taking limits and using the fundamental theorem of calculus we have \( w(x) = \int_0^1 w_i(y)dy \) does not depend on the path, and \( \lim_{h \to 0} \delta^h_i w(x) = w_i(x) \) pointwise for all \( x \in \Omega \). Thus \( \partial_i w = w_i \) in the sense of distributions as well. Therefore \( \partial_i(u - w) = 0 \) in the sense of distributions.

It remains to show that if \( u \in \mathcal{D}'(\Omega) \) and \( \partial_t u = 0 \) in the sense of distributions then \( u = \text{const} \), possibly after a change on a set of measure zero. Without loss of generality we may assume \( \Omega = (-a,a) \times \ldots \times (-a,a) \) is a cube centered at the origin. First we have the identity:

\[
\langle u, \varphi \rangle = 0, \quad \text{for all } \varphi \in C_0^\infty(\Omega) \text{ with } \int \varphi dx = 0 \text{ for some } i = 1, \ldots, n,
\]

which follows by writing \( \varphi = \partial_i \psi \), where \( \psi(x) = \int_{-\infty}^x \varphi dy \in C_0^\infty(\Omega) \) for such \( \varphi \). Now let \( \varphi_0 \in C_0^\infty((-a,a)) \) with \( \int \varphi_0 = 1 \). Then for any other \( \varphi \in C_0^\infty(\Omega) \) we get:

\[
\langle u, \varphi \rangle = \langle u, \varphi_0(x) \int \varphi dy \rangle,
\]

and by similar considerations for all \( 1 \leq k \leq n - 1 \):

\[
\langle u, \Pi_{i=1}^k \varphi_0(x^i) \int \varphi dy \rangle = \langle u, \Pi_{i=1}^{k+1} \varphi_0(x^i) \int \varphi dy \rangle.
\]

This implies:

\[
\langle u, \varphi \rangle = \int c_0 \varphi dx, \quad \text{where } c_0 = \langle u, \Pi_{i=1}^n \varphi_0(x^i) \rangle,
\]

In other words \( u = c_0 \) after a change on a set of measure zero.

\[\Box\]

Corollary 4.3. Let \( \Omega \subset \subset \mathbb{R}^n \) with \( \partial \Omega \) a \( C^{k+2} \) boundary. Let \( a^{ij} \in C^{k+1}(\overline{\Omega}) \) be elliptic, and assume \( b^i, c \in C^k(\overline{\Omega}) \). Suppose \( k > \frac{n}{2} \) and \( u \in H_0^1(\Omega) \) solves:

\[(17) \quad Lu = f, \quad \text{where } L = \partial_i a^{ij} \partial_j + b^i \partial_i + c,
\]

in the sense of distributions for some \( f \in H^k(\Omega) \). Then one has that \( u \in C^{2+l}(\overline{\Omega}) \) for any integer \( l \leq k - \frac{n}{2} - 1 \) when \( n \) is odd, and in fact \( u \) solves (17) pointwise in the classical sense (possibly after a change on a set of measure zero).

Remark 4.4 (Criticism of the method). While the previous method is sharp for general Sobolev functions \( f \), there is a defect if one considers \( u \in C^k(\overline{\Omega}) \). In this case one would expect to get \( u \in C^{k+1,\alpha}(\overline{\Omega}) \) for all \( 0 \leq \alpha < 1 \). Unfortunately just using \( f \in H^k(\Omega) \) loses roughly \( \frac{n}{2} \) derivatives because all we get from the above method is \( u \in C^{k+1-\frac{n}{2},\alpha}(\overline{\Omega}) \) for all \( 0 \leq \alpha < 1 \) (say when \( n \) is even).

There is another collection of bounds called Schauder estimates which are geared towards closing this gap. These are closely related to proving \( W^{k,p} \) bounds for elliptic equations (specifically for \( p = \infty \)). But this would require a few more lectures to cover!