I. NLS

A. Equation & conservation

We now consider the equation \( iu_t + \Delta u = F(u) \) where

\[ F = V'(i\mu_1)u \text{ for radian } V(\cdot). \]

Typically \( V = \lambda \frac{\partial}{\partial \lambda} S_\mu \)

so \( F = \lambda \mu_1 u \). The two main cases are \( \mu_1 \leq 0 \), focusing, \( \mu_1 > 0 \), defocusing.

1. Solutions to this equation (assuming they are smooth & well localized)

enjoy the two conservation laws:

1) Mass:
\[
\int_{\mathbb{R}^d} |u(x, t)|^2 \, dx = \text{const, which comes by integrating } \int_0^t \int_{\mathbb{R}^d} \text{Im} (\nabla |u|^2) \, dx \, dt
\]

2) Energy:
\[
\int_{\mathbb{R}^d} \left( \frac{1}{2} |u(x, t)|^2 + V(|u(x, t)|) \right) \, dx \text{ which comes by integrating } \int_0^t \int_{\mathbb{R}^d} \text{Re} (\nabla |u|^2) \, dx \, dt
\]

The \( S(u) = iu_t + \Delta u - F(\mu u) = 0 \) for solutions.

B. Linear Evolution and Strichartz

For NLS \( (P_{\mu}(\cdot)) \) estimates are key. Here is the main result we'll use:

Then: (non-endpoint Strichartz) let \( \mu u_t + \Delta u = F \) on \( \mathbb{R}^{d+1} \). Let \( \mu \geq 0 \)

and \( \mu \leq 0 \) be such that \( \frac{d}{d} + \frac{d}{2} = \frac{d}{2} \). Then one has the

space-time bound:
\[
\| u \|_{L^3(\mathbb{R}^d)} \lesssim C \| u_0 \|_{L^\infty(\mathbb{R}^d)} + \| F \|_{L^{8/3}(\mathbb{R}^d)}
\]

where \( (\mu, \delta) \) satisfies the same conditions as \( (\mu, \delta) \).

proof (sketch): This is similar to what we did for wave. First assume

\( F = 0 \), and prove \( \| e^{-it\Delta} u_0 \|_{L^3(\mathbb{R}^d)} \leq C \| u_0 \|_{L^\infty(\mathbb{R}^d)} \) by a TT

argument. By duality this becomes \( \| \int e^{it\omega - t^2/2} F(s) \, ds \|_{L^3(\mathbb{R}^d)} \leq C \| F \|_{L^{8/3}(\mathbb{R}^d)} \)

Then proved in the usual way by interpolation of

\[
\| e^{it\omega \Delta} F(s, \cdot) \|_{L^{\infty}} = \| F(s, \cdot) \|_{L^{\infty}} \quad \text{and} \quad \| e^{it\omega \Delta} F(s, \cdot) \|_{L^2} \leq \| F(s, \cdot) \|_{L^2}
\]

to get \( \| e^{it\omega \Delta} F(s, \cdot) \|_{L^3} \leq \frac{1}{1 + t^2/4} \| F(s, \cdot) \|_{L^3} \).
Then use HLS which says \( \frac{1}{1+\beta} + L^p \rightarrow L^p \) when \( r = \frac{2}{p} \) \( \left( \frac{1}{R_{\infty}} \L^p \rightarrow L^p \right) \).

Now \( r = \frac{1}{\beta} = \frac{d}{b} - \frac{d}{b} \) so the condition \( \frac{2}{p} + \frac{d}{b} = \frac{d}{b} \) and \( p > 2 \) follows.

Next, we get the dual Strichartz bound \( \| S_t e^{i(t-a) A} F_0 \|_{L^p} \leq \| F_0 \|_{L^p} \),

which combined with Strichartz gives \( \| S_t e^{i(t-a) A} F_0 \|_{L^p} \leq \| F_0 \|_{L^p} \).

Finally, we may conclude with:

\[ \text{Thm: (Christ-Kiselev) let } X, Y \text{ be Banach spaces, and } I \in \mathbb{R} 	ext{ on normal.} \]

\[ \text{let } k \in C^0(I; X \rightarrow X(I), Y)) \text{ and } L^p \text{ be such that:} \]

\[ \| S_t k(t, x) \|_{L^p(I; Y \rightarrow Y)} \leq C \| k \|_{L^p(I; X \rightarrow Y)} \].

Then one also has

\[ \| S_t k(t, x) \|_{L^p(I; Y \rightarrow Y)} \leq C \| k \|_{L^p(I; X \rightarrow Y)} \text{ for } \| k \|_{L^p(I; X \rightarrow Y)} \leq 1 \text{ and } \| k \|_{L^p(I; X \rightarrow Y)} \leq 1. \]

Remark: the condition \( pq \geq 2 \) needed in general.

II. \( \text{X} \subset \text{R}^P \) in the mass subcritical range.

A. Iteration space

Now consider the equation \( iu_t + \Delta u = -u \mu |u|^{p-1} u \)

with \( \mu(t, x) = \mu(t) \). We'll look for a solution with

where \( u \in C^0(I; L^p \cap L^r) \) solving the equation in the sense of

(weakly) distributions on \( (\gamma_1 \times \mathbb{R}^d) \).

To find a solution we'll look for a fixed point of

\[ I(u) = \frac{1}{\beta} \int_0^1 e^{i(t-s) A} F(s, u)(x) \mu_s(x) \, ds + \frac{i \mu}{2} u_0 \text{ in the space } \mathcal{S} \subset \mathcal{C}(C) \cap L^p \cap L^r \text{ when } \frac{2}{p} + \frac{d}{p} = \frac{d}{2}. \]

The key estimate in this case is

\[ \| I(u) - I(v) \|_{\mathcal{S}} \leq C \| u \|_{L^p} \| v \|_{L^p} \times \| u \|_{L^r} \times \| v \|_{L^r}. \]
Using \[ \left| \int u_1^{p_1} u - v_1^{p_1} v \right| \leq \mathcal{C} \left( \left\| u_1^{p_1} \right\|_{L^p(\Omega)} + \left\| v_1^{p_1} \right\|_{L^p(\Omega)} \right) \left\| u - v \right\|_{L^q(\Omega)} \] and Hölder's

we get \[ \left\| \mathcal{I}(u) - \mathcal{I}(v) \right\|_{L^p(\Omega)} \leq \mathcal{C} \left( \left\| u_1^{p_1} \right\|_{L^p(\Omega)} + \left\| v_1^{p_1} \right\|_{L^p(\Omega)} \right) \left\| u - v \right\|_{L^q(\Omega)} \].

The mean condition we need is \[ \frac{1}{p_1} \geq \frac{1}{p} \] at least \( p_1 \leq p \), which happens when \( \frac{1}{p} = \frac{1}{p_1} \). We also have \[ \frac{p}{p_1} = \frac{\frac{p}{p_1}}{p} \] so we need \[ p \leq 1 + \frac{1}{p_1} \] which is exactly the "mass (sup)-critical" condition. Now there are two cases

1. \( 1 \leq p < 1 + \frac{1}{p_1} \Rightarrow \left\| \mathcal{I}(u) - \mathcal{I}(v) \right\|_{L^p(\Omega)} \leq \mathcal{C} \left( \left\| u_1^{p_1} \right\|_{L^p(\Omega)} + \left\| v_1^{p_1} \right\|_{L^p(\Omega)} \right) \left\| u - v \right\|_{L^q(\Omega)} \] for some \( \mathcal{C} > 0 \). Then the map \( \mathcal{I} \) is a contraction on the ball \[ \left\| u_1^{p_1} \right\|_{L^p(\Omega)} \leq \mathcal{R} \] for \( T \leq T_\mathcal{R}(\mathcal{R}) \).

2. \( p = 1 + \frac{1}{p_1} \Rightarrow \left\| \mathcal{I}(u) - \mathcal{I}(v) \right\|_{L^p(\Omega)} \leq \mathcal{C} \left( \left\| u_1^{p_1} \right\|_{L^p(\Omega)} + \left\| v_1^{p_1} \right\|_{L^p(\Omega)} \right) \left\| u - v \right\|_{L^q(\Omega)} \] for all \( T \).

Using the conservation of mass the solution in 1) can be seen to be global in time. Also, when \( p = 3 \) and \( m = 1 \) (smooth \( F(u) \)) we can include to recover \( H^s \) bounds.