Topics - Picard & Existence for ODE, Gronwall & Stability, continuation principle, conserved quantities & global existence.

I. Basic ODE & Picard Theory

A. ODE and IVP

Let \( \Omega \subseteq \mathbb{R}^d \) be a domain, and \( F : \Omega \rightarrow \mathbb{R}^d \) be a function, \( F = (F_1, \ldots, F_d) \). Then

we want to solve \( \dot{x} = F(x), \quad x(0) = x_0 \).

B. Examples

1. Let \( \mathbb{R} \rightarrow \mathbb{R} \), solve \( \dot{x} = -x^2 \) (Van der Pol's Eq., \( s = e^{\alpha t} \))

Thus the equation becomes \( \frac{d^2 x}{dt^2} + (\alpha^2 - x^2) = 0 \).

If \( F = \frac{1}{2} m \dot{x}^2 + V(x) \), then \( \frac{d}{dt} E = 0 \) for a solution.

Example 2: Let \( G : \mathbb{R}^2 \rightarrow \mathbb{R} \), solve \( \dot{x} = -V(x) \). Then \( \int (\dot{x}^2 + V(x)) \, dt \)

Thus to a critical point of \( G \).

C. Picard Theory

Then let \( F : \Omega \rightarrow \mathbb{R}^d \) be \( C^1 \) (i.e., \( \frac{\partial F}{\partial x} \)) (\( C_k \) all \( \Omega \)), and for each \( x_0 \in \Omega \)

on \( B(0,\alpha) \) such that \( x = F(x) \) has a solution for \( \| x_0 \| < \alpha \). This solution is unique.

Proof: We only care about a constant end if \( x \in B(0,\alpha) \), so WLOG assume \( F \) is globally defined and \( \| F \|_{\text{lip}} = M \).

First set up an integral equation \( x(t) = x_0 + \int_0^t F(x(s)) \, ds \), where \( \frac{d}{dt} [x(s)] = F(x(s)) \).

is defined for \( s \in [0, t] \), with \( s \in (y(x), t) \), \( h(t) = \int_0^t F(x(s)) \, ds \).

Thus \( \| s \|_{\text{lip}} \leq M \tau \| x_0 \|_{\text{lip}} \). \( \| x \|_{\text{lip}} \leq M \tau \| x_0 \|_{\text{lip}} \).

Remark 1: This also applies to \( \dot{x} = F(t, x) \) by setting \( y(t) = (1, x(t)), \quad G(y) = (1, F(t, x)) \), \( \text{then } \dot{y} = F(y), \quad \text{y}(0) = (1, x) \).

Thus the original.

Remark 2: \( \dot{y} = F(y), \quad y(0) = (1, x) \) has two solutions, \( x(0), \quad \frac{1}{t} \log \| y(t) \| \).
Remark: In general, the solution is only loud even if $f$ is $g$-globally loud. For example, $x(t) = x_0 e^{t/2}$.

II. Stability

Let $\mathcal{F}(x) = f(x) + g(x)$, $\mathcal{F}(x_0) = x_0$. Call the "flow map." We'd like to understand the continuity properties of this map for fixed $t$.

A. Gronwall's Estimate

Then: Let $\alpha(t)$ be a solution such that $\alpha_0 = 0$, $t_0 = 0$, and $\alpha(t) \leq A + \int_{t_0}^{t} g(s) ds$ for some $A > 0$.

Then $\alpha(t) \leq A e^{\int_{t_0}^{t} g(s) ds}$, $t_0 \leq t$.

B. Stability Estimate

We now apply this to $x(t) = x_0 e^{t/2}$ is Lipschitz for fixed $t$.

Then: Let $F: \mathbb{R}^d \to \mathbb{R}^d$ be a (locally) Lipschitz vector field with $\|F(\mathcal{C}(x))\| = M$ some $k \in \Omega$.

Then for $x, y \in \mathcal{C}(x)$, $\|F(x) - F(y)\| \leq M \|x - y\|$ for all $x, y, t_0 \leq t$.

III. Conservation Principle & Conserved quantities

A. General Conservation Principle

Then: Let $F: \mathbb{R}^d \to \mathbb{R}^d$ be locally Lipschitz, and $x(0) = x_0$. Then if $\mathcal{I}(x_0(t))$ is the maximum interval of existence of $x(t)$, $x(t) = x_0$, we must have at least one of:

1. $t \geq 0$

2. $t < \infty$ and $\lim_{t \to \infty} x(t) = x_0$. 
3. If \( T \) is a monotonous function, and \( \text{dist}(x, y) = 0 \), for some sequence of times, then there exists a sequence of points in \( T \) with \( \text{dist}(x, y) = 0 \).

For some \( x \) and \( y \), and some \( t \), the function \( T \) is continuous.

3. The use of hankel quantities

Let \( F : \mathbb{R}^d \rightarrow \mathbb{R} \) be a \( C^1 \) function such that \( F(x) \) is finite.

Then \( x(1) \)-ball is the smallest \( x \) such that \( F(x) \) is finite.

If \( x \) is a \( C^1 \) function, then \( x(1) \)-ball is the smallest \( x \) such that \( F(x) \) is finite.

Let \( x \) be a ball in \( \mathbb{R}^d \). Then \( x(1) \)-ball is the smallest \( x \) such that \( F(x) \) is finite.

Now assume \( x(1) \)-ball is finite. Then \( x(1) \)-ball is the smallest \( x \) such that \( F(x) \) is finite.

If \( x(1) \)-ball did not exist, then \( x(1) \)-ball would not be finite, so \( F(x(1)) = \infty \).

Theorem: If \( x(1) \)-ball is finite, then \( x(1) \)-ball is the smallest \( x \) such that \( F(x) \) is finite.

Finally, \( x(1) \)-ball would not be finite, so \( F(x(1)) = \infty \).

Theorem: If \( x(1) \)-ball is finite, then \( x(1) \)-ball is the smallest \( x \) such that \( F(x) \) is finite.

Exercise