Overview: Geometrical Optics for wave, local energy estimates

I. Asymptotic solutions to the wave equation

A. The equation

* In this lecture we’ll start to compute solutions to \( P(x,y) = \Delta_y \)

where \( \Delta_y = \sum_{i,j} \partial_i \xi_i \partial_j \xi_j \), \( \xi_i \) is a Riemannian metric.

B. The ansatz

* Well begin by constructing solutions in the form \( \phi = a_x e^{i\lambda x} \), where
  
\( a_x = a_x(t;x) \), and \( u(x) \) does not depend on \( \lambda \)

* Plugging in we get \( \Delta_y P(x) = \lambda^2 (15u(x) - u_t) + 2i \lambda (u_t \partial_t a_x - \langle u, u(\partial_x a_x) + \frac{1}{2} \partial_x a_x \rangle \partial_x \)

* As \( \lambda \to \infty \) each term must be eliminated separately.

C. The eikonal equation

* First we solve \( u_t^2 = \Delta_x u \). Setting \( u^2 = t + g(x) \), the boils down to solving \( \Delta g = 0 \), which is given by some distance function.

* Note that if

D. The transport equations

* To solve the remaining parts of the system we’ll expand \( a_x = \sum_{\ell=0} a^{(\ell)}(\xi) \).

Then we set:

\[ X_{a_0} = \frac{1}{\ell} \partial a_0 \]
\[ X_{a_{\ell+1}} = \frac{1}{\ell} \partial a_{\ell+1} + \chi_{\ell+1} \]

where \( \chi = -\partial u \), \( \chi = \partial_x \Delta_x g = \partial_t + g \partial \partial_x g \).

Recall that \( \bar{\xi} \partial \bar{g} = \Delta \), and \( \bar{\xi} \partial g^2 = \frac{1}{2} \) where \( \bar{\xi} + \Gamma_{\bar{\xi} \xi} \bar{\xi} = 0 \)

are the geodesics normal to \( \bar{g} = \text{const.} \)

* This shows \( X = \chi_0 + \chi_{\ell} \) transports \( a_x \) along geodesics \( \bar{\xi}(\ell) \).
\[
\frac{d}{dt} a(t, x(t)) = \partial_t a + i \cdot \partial_x a = 0. \quad \text{So} \quad a(t, x(t)) = a_0(x_0),
\]

where \( x_1 = x_0 \).

E. Error term

* From the construction, we see

\[\Phi_h = \kappa \cdot \Phi_{ho} \cdot e^{\lambda t}.\]

Thus \( \| \Phi_h \|_{H^1} \approx \lambda^{5-N} \), while \( \| \Phi_h \|_{H^1} \approx \lambda^5 \).

III. Local energy Estimates for Wave

A. A formula from geometry

\* Let \( \lambda I \lambda \mu \lambda \), and assume \( 0 < T \leq 1 \) near \( 2S_{ho} = \{ z = 0 \} \). We'll set \( \Omega = \{ z < t \} \).

\* Consider \( \frac{d}{dt} \partial_t \Phi_h \cdot \partial_x \Phi_h \), where \( \partial_t \Phi_h = \mu \partial_x \Phi_h \).

\* Locally close to \( 2S_{ho} \), we can change conts \( \{ z, \lambda \} \), so \( g = \lambda^2 + \mu \partial_x \Phi_h \partial_x \Phi_h \), \( z = 1, \ldots, n \).

Then \( \partial_t \Phi_h = \partial_x \Phi_h \), \( \partial_x \Phi_h = 0 \). \( S_{ho} \) is the area \( z = \text{cont} \).

\* Using this we have \( \frac{d}{dt} \partial_t \Phi_h \cdot \partial_x \Phi_h \cdot \partial_x \partial_t \Phi_h \cdot \partial_x \Phi_h = -2 \partial_x \Phi_h \cdot \partial_x \Phi_h \).

B. Local energy

\* Now let \( S_{ho} \) be as above. Suppose \( \Phi_h \equiv 0, \Phi = \mu \cdot \Phi_0 \).

\* Consider \( 
\int_0^T \int_{S_{ho}} \left( \partial_t \Phi_h \cdot \partial_x \Phi_h \right) \partial_x \partial_t \Phi_h \cdot \partial_x \Phi_h dt \)

\[= E_{\Phi_h} (I)_{t=T}^{t=0} \int_0^T \int_{S_{ho}} \left( \partial_t \Phi_h \cdot \partial_x \Phi_h \right) \partial_x \partial_t \Phi_h \cdot \partial_x \Phi_h dt \int_{S_{ho}} \left( \partial_t \Phi_h \cdot \partial_x \Phi_h \right) \partial_x \partial_t \Phi_h \cdot \partial_x \Phi_h dt dt \]

\* The last two terms combine by splitting \( 1 \leq \lambda \leq 2 \), so \( E_{\Phi_h} (I) \leq E_{\Phi_0} (I) \), \( L \leq 2 \).

\* In particular this shows \( E_{\Phi_0} (I) \leq E_{\Phi_0} (I) \).

\[\text{ illustrating a diagram} \]

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