I. Some global theory for nonlinear waves

A. Review of lifespan for algebraic powers.

Now we look at a general class of nonlinear problems

\[ \Delta \Phi = N(\Phi, \partial_x \Phi) \quad \text{where} \quad N \text{ is at least quadratic in } (\Phi, \partial_x \Phi). \]

For wave maps \( N = \frac{\partial_x^2 \Phi}{\partial_t \Phi} \), \( m^2 \ll \delta(1, \ldots, \zeta). \)

In this case we'll use enough regularity to get into \( N \in \mathbb{H}^s \), so data in \( H^3 \mathbb{H}^{s+1} \) for \( s > \frac{d+1}{2} \). In fact to make the proofs go a bit easier we'll assume \( s \geq \frac{d}{2} + 2 \).

Assuming \( N \in \mathbb{C} \) the main term we need to contend with here 3

\[ \left\| \int L_x \frac{\partial_x^2 \Phi}{\partial_t \Phi} \frac{\partial_x^2 \Phi}{\partial_t \Phi} \right\|_{L^2} \leq \text{constant} \]

where \( \sum_{k} s_k = 0 \) The key observation now is that for all \( s_k \in \mathbb{R} \), have \( s_k \leq \frac{d+3}{2} \), so \( s + 1 < \frac{d+2}{2} \).

Then, using \( H^s \subset L^2 \) we get \( \|N(\Phi, \partial_x \Phi)\|_{H^s} \leq C_0 \| \Phi \|_{H^s}^2 \)

which gives a lifespan of \( O(\| \Phi \|_{H^s}^2) \).

B. Riccati: Models of Global Behavior

First look at some global models where decay helps to mitigate finite time concentration.

\[ \phi_x = ax \phi^2, \quad \text{say } a > 0 \text{ and } \phi(0) > 0. \]

Then with \( \mu = \frac{1}{a} \) the equation becomes \( \mu \phi_x = -a, \) so \( \mu \phi_x = \phi - S \phi \) where \( \mu = \frac{1}{a} \).

Now \( \mu < 0 \) until \( \phi = 0 \) so there are three cases:

1) \( S_0 < \frac{1}{\mu} \) \quad \text{global existence.}

2) \( S_0 \text{, } \mu \) \quad \Rightarrow \text{ Blowup at } b \mu t_0

3) \( S_0 \text{, } \mu \) \quad \text{leading } S_0 \text{, } \mu \text{, } \mu \text{, } \text{ Blowup at } b \mu t_0 \).
Note that in case 1) above \( \phi = \frac{1}{\phi^{2} p_{0}} \) is fixed, so for data \( \phi(t) \) there is global existence, while for \( \phi(t) \) there is blow-up in finite time.

Another interesting thing to point out is the marginal case \( e^{-\phi} \) as \( \phi \to 0 \).

The \( \phi \to 0 \) when \( \phi \to 0 \) and \( \phi^{2} \) solutions last until \( \phi^{2} = \frac{1}{\phi^{2}} \), or \( e^{-\phi/\phi} \). This is sometimes called "almost global existence!"

C. Global existence via vector-fields

To handle global problems we use a simple bootstrapping argument:

Let \( A(t) \) be a continuous function defined on an interval \( I \) such that

\[
A(t) \leq A_{0} + C A^{2}(t), \quad \text{some} \quad C > 0, \quad p > 1. \quad \text{Then} \quad A(t) < \frac{1}{e^{p-1} C^{2}} + \frac{1}{e^{p-1} C^{2}} \quad \text{and hence}
\]

exists to \( I \) with \( A(t) \leq A_{0} \) and has \( A(t) \leq A_{0} \) all \( t \in I \).

Proof: let \( I \) be all points with \( A(t) \leq A_{0} \). Then \( t \in I \), and by continuity \( \hat{A} \) is closed in \( I \). On the other hand \( \hat{A} \) is closed in \( I \), we also have \( A(t) \leq A_{0} + C A^{2}(t) \) and \( C A^{2}(t) \leq C(2A_{0})^{2} < C(\frac{1}{p})^{2} \).

Thus \( A(t) < A_{0} + C A^{2}(t) = A(t) \). Thus \( A(t) \) is closed in \( I \) by continuity.

Since \( \hat{A} \) is non-empty \( \cap \) both \( \in \) closed in \( I \) we must have \( I = I \).

Remark: Nonlinear bounds don't do much good when things are large. For example

\[ I = [0,1], \quad A(t) = \frac{1}{t^{2}} \quad \text{satisfies} \quad A(t) \leq e^{2}(t) \quad \text{all} \ t \in I. \]

We now have the following global result:

Then let \( \phi = N(0,\phi) \) be at least \[ \begin{cases} \text{small} & e^{2} \geq \phi \geq e^{2} \text{ or } \phi \text{ for } \end{cases} \]

\[ \text{small} \text{ data in } S(\mathbb{R}^{d}) \text{ there is a global solution.} \]
The method is based on a-priori bounds in $S(\Omega_1)$ defined by
$$
\| f \|_{S(\Omega_1)}^2 = \sum_{\ell=0}^{s_0} \| \nabla^\ell f \|_{L^2(\Omega_1)}^2, \quad \text{and} \quad \| f \|_{H^s(\Omega_1)}^2 = \sum_{\ell=0}^{s_0} \| \nabla^\ell f \|_{L^2(\Omega_1)}^2,
$$
where $x \in \partial \Omega_1, \Omega_1, \Omega_2, \Omega_3$. From the last lecture, we had the two bounds:

1) $Z_{\ell=0} \| X^\ell f \|_{L^2(\Omega_1)}^2 \leq C_s$ for $\| f \|_{S(\Omega_1)}^2$, where $\lambda_s = \left( \frac{\mu}{\omega} \right)^{s+1}$.

2) $\| f \|_{S(\Omega_1)}^2 \leq C_k \left( \| f \|_{S(\Omega_1)}^2 + \| \nabla f \|_{S(\Omega_1)}^2 \right)$

In order to bootstrap estimates, we need to show $\| f \|_{S(\Omega_1)}^2 \leq C_k \| f \|_{S(\Omega_1)}^2$, where $p$ is the total degree of $N(\Omega_1)$ as $x \to x_0$.

Choosing $k > d+1$, we can have at most $\| \nabla^d f \|_{L^2(\Omega_1)}^2$, deriving on the very frequency $k$ term. On the other hand, $\| f \|_{S(\Omega_1)}^2 \leq \left( \frac{\mu}{\omega} \right)^{s+1}$ for $d \to \infty$.

So we can balance with 1).

* In a similar way, we can also prove 2.

Then: (John's Almost Global Existence Result)

Let $\Omega = N(\Omega_1)$ where $N$ is quadratic. Then for $\| f \|_{S(\Omega_1)}^2 \leq \varepsilon_0$, we can find a solution with lifespan $T_0 = \varepsilon_0$ for some $\varepsilon_0 > 0$.

Then: (John's Blowup Result) For each $\varepsilon_0 > 0$, an open set of data supported in $\Omega$ with $\| f \|_{S(\Omega_1)}^2 \leq \varepsilon_0$ such that the solution to $\Omega = \varepsilon_0 T_0$.

We'll just prove this for radial solution. The equation becomes
\[ \frac{1}{\rho} \frac{\partial}{\partial r} \left( \rho \frac{\partial \psi}{\partial r} \right) = \frac{1}{4} \frac{\partial^2 \psi}{\partial \xi^2} \] \quad \rho = \frac{3}{8} + \frac{3}{8} \psi \quad \frac{\partial \psi}{\partial \xi} = 0 \quad \frac{\partial \psi}{\partial \eta} = 0 \quad \frac{\partial \psi}{\partial \xi} = 0 \quad \frac{\partial \psi}{\partial \eta} = 0 \quad \frac{\partial \psi}{\partial \eta} = 0

In the outgoing region, we have the system
\[ \frac{\partial \psi}{\partial \xi} = \frac{1}{4} \frac{\partial^2 \psi}{\partial \xi^2} \quad \frac{\partial \psi}{\partial \eta} = \frac{1}{4} \frac{\partial^2 \psi}{\partial \eta^2} \quad \frac{\partial \psi}{\partial \xi} = \frac{1}{4} \frac{\partial^2 \psi}{\partial \xi \partial \eta} \quad \frac{\partial \psi}{\partial \eta} = \frac{1}{4} \frac{\partial^2 \psi}{\partial \eta \partial \xi} \]

If we assume \( \psi \to 0 \) at \( t = 0 \), the condition propagates because the mixed terms \( \frac{1}{4} \frac{\partial^2 \psi}{\partial \xi \partial \eta} \) are integrating factors which cannot change the sign. Thus \( \frac{\partial \psi}{\partial \xi} = \frac{1}{4} \frac{\partial^2 \psi}{\partial \xi^2} \) along outgoing rays, which give blow-up by time \( T \to 0^+ \), when \( t = \pi(r) \).