Outline: Final wave, basic LWP for non-linear problems

I. Some non-linear problems

* We’ll now look at a number of problems based on the wave equation.

\[ D_u^2 \phi - c^2 \Delta \phi = 0. \]

A. Semilinear problems:

\[ D_u^2 \phi = \lambda \phi^p \quad \text{where} \quad \lambda \in \mathbb{R} \quad \text{and} \quad p \in \mathbb{N}. \]

In this case one has the energy:

\[ E[\phi(t)] = \frac{1}{2} \int \left( c^2 \Delta \phi^2 + \frac{1}{p+1} \phi^{p+1} \right) dx. \]

* We call this equation “focusing” if \( \lambda > 0 \), and “defocusing” if \( \lambda < 0 \).

* Note that in the latter case \( \| \phi(t) \|_{H^1}^2 \leq 2E \), so one gets an a-priori bound on \( \| \phi(t) \|_{H^1} \).

B. Semilinear DNLW problems

* These include problems like:

\[ D_u^2 \phi = | \phi |^2 \phi, \quad D_u^2 \phi = \lambda \phi^4 \phi, \quad D_u^2 \phi = \lambda \phi^2 (\phi^2 + 1)^2 \]

where \( \lambda(t, \phi) \) is at least linear in \( \phi, \psi \).

* A basic example here are the wave maps \( \phi: \mathbb{R}^n \to (M, g) \)

when in local coordinates:

\[ D^2 \phi = G^{ij}(\phi) \partial_\theta^i \phi \partial_\theta^j \eta^{\theta^i} \]

where \( \eta^{\theta^i} \) indexing \( (1, 1, \ldots, n) \) is the Minkowski metric.

* These generalize both the geodesic case \( c_0 : \mathbb{R}^2 \to (S^2, g) \)

and the equation for harmonic maps: \( \Delta \phi = -\Gamma_{\theta \theta}^{\phi} \phi_{\theta} \phi_{\theta} \).

* These conserve the geometric energy:

\[ E[\phi(t)] = \frac{1}{2} \int \left( c^2 \Delta \phi^2 + \frac{1}{p+1} \phi^{p+1} \right) dx. \]

C. Quasilinear Equations
These are NLW of the form $\Box_{\text{Euclid}} \phi = N(\partial_t, \partial_x \phi)$

where $N(\partial_t, \partial_x \phi)$ is a DNLS term (at least quadratic),
and $g = g(\phi)$.

Bose example has the Einstein equations, where $h_{\mu \nu}$ is a
Lorentzian metric, where $\text{Ric}(h) = 0$. Then in coordinates $\partial_h x^2 = 0$
we have $\partial_h h_{\mu \nu} = \mathcal{N}(h, \partial_t h)$.

II. Local Well Posedness

A. Overview

1) Use energy estimates to construct local classical solutions $u(t, x) =_\text{a priori} C^1(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}^d)$

2) Show the solution map is Lipschitz.

3) Take limits to get “strong” solutions.

4) Use energy to propagate higher norms.

B. Picard Iteration

$$\phi_0 - \partial_0 \phi = -F'(\phi) = G(\phi), \quad E = \frac{1}{2} \int \frac{\partial_x^2 \phi}{\partial^2_x} + \int \frac{\partial \phi}{\partial_x} \partial_x \phi = \int \phi \, dx.$$
\[
\begin{align*}
\| \psi_{n} - \phi_{n} \|_{S_{n}} & \leq C_{T} \left( \| (\psi_{n}, \psi_{n})_{H_{0}} \|_{L_{2}}^{r} + \sum_{i} Q (\| (\psi_{n}, \psi_{n})_{H_{0}} \|_{L_{2}}^{r}) \cdot \| (\psi_{n}, \psi_{n})_{H_{0}} \|_{L_{2}}^{r} \right) \\
& \Rightarrow \| \psi_{n} - \phi_{n} \|_{S_{n}} \leq C_{T} \| (\psi_{n}, \psi_{n})_{H_{0}} \|_{L_{2}}^{r} \cdot \sum_{i} Q (\| (\psi_{n}, \psi_{n})_{H_{0}} \|_{L_{2}}^{r}) \cdot \| (\psi_{n}, \psi_{n})_{H_{0}} \|_{L_{2}}^{r}.
\end{align*}
\]

Finally, we also get bounds on the iteration

\[
\| \psi_{n} - \phi_{n} \|_{S_{n}} \leq C_{T} \| (\psi_{n}, \psi_{n})_{H_{0}} \|_{L_{2}}^{r} \cdot \sum_{i} Q (\| (\psi_{n}, \psi_{n})_{H_{0}} \|_{L_{2}}^{r}) \cdot \| (\psi_{n}, \psi_{n})_{H_{0}} \|_{L_{2}}^{r}.
\]

The important thing here is $T \leq T_{0}$ ($0, 1/\theta, 1/\theta_{1}$).

The second important thing is we needed $\theta > 1$. So $\theta > 1$ even when $d = 2$. 