I. Global Regularity for null forms

A. Scalar null form equation

In this section, we now look at quadratic null forms which have only “good” interactions.

**Example:** Let $\phi = 0$ with $\phi^{1}=1$. Let $\psi$ solve $\phi^{1}=\psi^{1}$. Then

$0 = \Delta \psi = \phi^{1}(\phi^{1} + \phi^{2}\phi^{1})$. Thus $\psi$ solves $\Delta \psi = -\phi^{2}\phi^{1}\phi^{1} = N(\phi^{1})$.

Conversely, any solution to this equation gives $\phi = 0$ when $\phi^{1}=\psi^{1}$.

Thus $\Delta \psi = N(\phi^{1})$ always has a global solution.

B. Klainerman's Act for null form systems

**Definition:** A quadratic null form $N^{ab}$ is called a “null form” if $N(\phi,\phi) = 0$ when $\phi^{a}\phi^{b} = 0$.

**Example:** $N^{ab} = \phi^{a}\phi^{b}$, $N^{ab} = -N^{ba}$. Linear combinations of these are a basis.

- **Lemma:** If $N^{ab}$ is a null form then $\Delta N^{ab} = 4\phi^{c}N^{bc} \leq C \left( (\phi^{1}+\phi^{1})^{2} + (\phi^{2}+\phi^{2})^{2} \right)$

  where $\phi \in \mathbb{R}$, $t \in \mathbb{R}_+$.

- **Proof:** Letting $\phi^{1}, \phi^{2}, \phi^{3}$ be a (local) basis of the tangent space, $N^{ab} \phi^{a}\phi^{b} = N^{\ell\delta} \phi^{\ell}\phi^{\delta}$

  $+ N^{\ell\delta} \phi^{\ell}\phi^{\delta} + N^{\ell\delta} \phi^{\ell}\phi^{\delta} + N^{\ell\delta} \phi^{\ell}\phi^{\delta}$. Note that $N^{11} = N^{22}$.

- **Lemma:** Let $X = \phi^{a}\phi^{a}$ be a vector-field. Then $X(N(\phi,\phi)) = L_{X}N(\phi^{1},\phi^{1}) + N(N(\phi,\phi))$.

  Hence $L_{X}N^{ab} = X(N^{ab}) = 2\phi^{a}N^{ab} - 2\phi^{b}N^{ab}$.

Finally, if $N$ is a null form and $X \Phi = 0$ for $\Phi \in \{\phi^{1}, \phi^{2}, \phi^{3}\}$, then $L_{X}N$

is also a null form.

**Proof:** The first calculation follows from the Leibniz rule and $\phi^{a}d\phi^{a}=dt$.
For the second we compute $\mathfrak{L}(f_5) = X(N(f_{11})) - N(f_{11}) - N(1,t,\omega)$

when $T$ is any solution of $f_5 = 1_{(8,\omega)}$. Here $\gamma, \delta_\Omega = 0$, we can extend $\gamma$ by Lie transport $\mathfrak{L}_{\xi} \gamma = 0$. Thus $(2,\rho,\gamma) = 0$. This gives $X(N(f_{11})) = 0$

so we are done.

Then: (Kahler's Global Existence) there exists $\omega_0$ such that for $\|\psi(t)\|_{\omega_0} \leq 0$

one has a global classical solution to the system $\mathfrak{R} \psi = N^i(\psi)$ where

$N^i$ we will have is $\psi = (\psi^1, \ldots, \psi^n)$.

\textbf{pf:} The key modification here is that we add some null energy estimates in the

region $\mathbb{K} \times (0, \infty)$. Thus $\|\psi(t)\|_{\omega_0} = \sum_1^{\infty} \|\psi(t)\|_{\omega_0} + \sum_1^{\infty} \|\psi(t)\|_{\omega_0}$

where $(\psi(t))$ are characteristic across $\psi = 0$.

The second term comes from the null energy estimates

$$\sum_1^{\infty} \|\psi(t)\|_{\omega_0} \leq C(E[\psi(t)]) + \sum_1^{\infty} \|\psi(t)\|_{\omega_0}$$

as usual. In fact we can just trade this for $\|\psi(t)\|_{\omega_0} = \sum_1^{\infty} \|\psi(t)\|_{\omega_0}$.

Now our goal now is to prove the "bilinear estimates" $\|\psi(t)\|_{\omega_0} = \sum_1^{\infty} \|\psi(t)\|_{\omega_0}$ for scales $\psi \psi$, when $N$ is a null form. Using the Leibniz rule we may assume its $\|\psi(t)\|_{\omega_0} = \sum_1^{\infty} \|\psi(t)\|_{\omega_0}$ when $N = \psi \psi$ (possibly different $\psi$). Now there are two cases:

1) $\|\psi(t)\|_{\omega_0} = 1_{(8,\omega)}$, use $\psi = \frac{1}{2} x + \frac{1}{2} x^2$ where $x \in \mathbb{K}$, $\omega_0, \delta_\Omega, \delta_\omega$

Then $1_{(8,\omega)} + \frac{1}{2} x^2 + \frac{1}{2} x^4 + 1_{(8,\omega)}$ and $\frac{1}{2} x^2 + \frac{1}{4} x^4 + \delta_\Omega + \delta_\omega$. $\frac{1}{2} x^2 + \frac{1}{4} x^4 \leq 1_{(8,\omega)}$.

This is enough to integrate in $f^2(\psi)$ vs $1_{(8,\omega)}$.

2) $\|\psi(t)\|_{\omega_0} = 1_{(8,\omega)}$. In this case use $1_{(8,\omega)} + \frac{1}{2} x^2 + \frac{1}{4} x^4 \leq 1_{(8,\omega)}$ and null energy.