I. Some properties of the FT on $L^p$.

Definition: If $u \in S'(\mathbb{R}^n)$ is a tempered distribution we define $U$ via the formula

$$U(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} u(x) e^{-i\xi \cdot x} \, dx,$$

for all $f \in S(\mathbb{R}^n)$. We call this the "inverse Fourier transform".

A general, but somewhat abstract, result about the computation of Fourier transforms and the Fourier inversion formula is the following:

Proposition: Let $u \in S'(\mathbb{R}^n)$. Then $\hat{\hat{u}} = u$. In addition, if $u \in L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and $\hat{u} \in S'(\mathbb{R}^n)$, then $\hat{\hat{\hat{u}}} = u$.

In addition, if we write

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \, u(x) \, dx,$$

then $\hat{\hat{u}} - \hat{u}$ is in $L^p(\mathbb{R}^n)$.

Proof: The abstract inversion formula follows from computing

$$\hat{\hat{u}}(\xi) = \int_{\mathbb{R}^n} e^{i\xi \cdot x} \, \hat{u}(x) \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\xi \cdot x} \, e^{-i\eta \cdot x} \, u(x) \, dx \, d\eta = u(\xi),$$

all $\xi \in S(\mathbb{R}^n)$, so $\hat{\hat{u}} = u$ (the kernel of a distribution is unique). Suppose $\hat{u} \in S'(\mathbb{R}^n)$ with $\hat{\hat{u}} = u$, then by Hausdorff-Young

$$\|\hat{u}\|_{L^p} \leq \|u\|_{L^q} = \|u\|_{L^p}.$$ 

The same result for $\hat{u}$ replaced by $\hat{\hat{u}}$ follows similarly.

Just like before we have the following list of identities for the FT on $L^p$.

Proposition: Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ be functions and let $\hat{f}, \hat{g}$ be their Fourier transforms in the sense of distributions. Then:

1) $\hat{\hat{f}}(\xi) = f(\xi)$ and $\hat{\hat{g}}(\xi) = g(\xi)$ for all $\xi \in \mathbb{R}^n$.

2) $\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$ when $1/p + 1/q = 1$.

3) $\hat{fg} = \hat{f} \hat{g}$ and $(\hat{f}, g^*) = (f^*, \hat{g})$ when $1/p + 1/q = 1$. 


4) If \( f \in L^p(m) \) for \( 1 \leq p \leq 2 \) and \( \mu \leq 1 \) then \( \left( \int |f|^\mu \right)^{1/\mu} \leq C_{\mu, p} \left( \int |f|^p \right)^{1/2} \).

5) If \( p = 2 \) then \( (f, g)_{L^2} = (f, g)_{L^2} \).

**Proof:** The proofs follow easily from the corresponding statements for \( SL^p(r) \) and density. Note that the conditions on \( p, \mu \) in 3)-6) are so Hoelder-Yong is satisfied. The constants in 3) come from \( (\int |f|^p)^{1/2} = (\int |f|^p)^{1/2} = (\int |f|^p)^{1/2} \). Part 4) follows by taking the FT of both sides.

Now we can prove the following thing about Fourier inversion.

**Theorem:** Let \( f \in L^p(m) \) with \( 1 \leq p \leq 2 \). Let \( \mu \in SL^p(m) \) such that \( \mu(a) = 1 \). Set

\[
f_\mu(x) = \frac{1}{\mu(m)} \int e^{\overline{x} \cdot y} \mu(\overline{y}) f(y) dy.\]

Then \( f \rightarrow \mu \rightarrow \mathfrak{F} \) for all \( x \in \mathbb{R}^d \).

**Proof:** \( \mu(\overline{y}) \in SL^p(m) \), so \( f_\mu(\overline{y}) = (\mu(\overline{y}) \mu^{1/2})^{1/2} \). We have \( \mu(\overline{y}) \mu^{1/2} \leq C \mu(\overline{y}) \leq C \mu(\overline{y}) \). Since \( \mu \in SL^p(m) \), it is easy to check \( \mu(\overline{y}) \leq C \mu(\overline{y}) \mu^{1/2} \) and \( \sum \mu(\overline{y}) = 1 \). Thus \( f_\mu(\overline{y}) = \mu(\overline{y}) \) is an approximate identity and the result follows.

**Remark:** By splitting functions, the previous theorem also shows that for \( f \in L^p(m) \)

one has \( \left( \int |f|^{2p} \right)^{1/p} = \left( \int |f|^p \right)^{1/2} \). For any \( \mu \in SL^p(m) \) with \( \mu(a) = 1 \).

A typical example is \( f_\mu(x) = \frac{1}{\mu(m)} \int e^{\overline{x} \cdot y} \mu(\overline{y}) f(y) dy \) for all \( f \in L^p(m) \).

II. The Uncertainty Principle

We start with the most basic uncertainty principle which states that if
\[ \hat{\mathbf{f}} \] is concentrated on a box of size \( \lambda_1 \times \ldots \times \lambda_n \), then the support of \( \hat{\mathbf{f}} \) cannot be compressed further than boxes of size \( \lambda_1' \times \ldots \times \lambda_n' \).

Then: (Heisenberg's Uncertainty Relation) Let \( f \in L^2(\mathbb{R}^n) \), then for \( (\mathbf{x}, \mathbf{\xi}) \in \mathbb{R}^{2n} \) one has
\[
||\hat{f}||_2 \leq \frac{2}{\pi} \left( \sum_{k=1}^{n} ||x_k - \xi_k|| \right)^{1/2} \left( \frac{2\pi}{n} \right)^{1/4}. The constant is sharp in the sense that there is equality for \( f = |A|^k e^{iAx \cdot \xi} \) when \( A = (\lambda_1', \ldots, \lambda_n') \) is a positive definite diagonal matrix.

Remark: By the Cauchy-Schwarz inequality one has
\[
||\hat{f}||_2 \leq \frac{2}{\pi} \left( \sum_{k=1}^{n} ||x_k - \xi_k|| \right)^{1/2} \left( \frac{2\pi}{n} \right)^{1/4}. \]
which in general is a less precise form.

pf: First compute \( \text{Im} \left( D_{x_k} \hat{f}, x_k' \right) = \frac{1}{2i} \left( (D_{x_k} f, x_k') - (x_k f, D_{x_k} x_k') \right) \)
\[
\Rightarrow \frac{1}{2} \frac{d}{dx_k} \left( f(x) \hat{f}(\xi) \right) = \frac{1}{2} \frac{d}{d\xi_k} \left( f(x) \hat{f}(\xi) \right). \]
Thus \( \frac{1}{2} \frac{d}{dx_k} \left( f(x) \hat{f}(\xi) \right) = 2 \left( \sum_{k=1}^{n} ||x_k - \xi_k|| \right)^{1/2} \frac{d}{d\xi_k} \left( f(x) \hat{f}(\xi) \right) \). Summing on \( k=1, \ldots, n \) yields
\[
\frac{1}{2} \frac{d}{dx} \left( f(x) \hat{f}(\xi) \right) = 2 \left( \sum_{k=1}^{n} ||x_k - \xi_k|| \right)^{1/2} \frac{d}{d\xi} \left( f(x) \hat{f}(\xi) \right). \]
Applying this estimate to \( e^{ix \cdot \xi} \) yields the desired result by noting \( e^{ix \cdot \xi} \hat{f}(\xi) = e^{i(\xi \cdot \xi_0)} \hat{f}(\xi + \xi_0) \).

To see the optimality of the Gaussian note that \( e^{i(\xi \cdot \xi_0)} \hat{f}(\xi) = e^{i(\xi \cdot (\xi_0 - \xi))} \hat{f}(\xi) \), so with \( f(x) = e^{ix \cdot \xi_0} e^{-1/2 (A(x-x_0))^2} \) then \( \hat{f} = (2\pi)^{n/2} e^{-1/2 (A^{-1}(\xi-x_0))^2} \) which is a Gaussian function.

By change of variables we compute \( \frac{1}{2} \frac{d}{d\xi} \left( f(x) \hat{f}(\xi) \right) = \int e^{-i(\xi \cdot \xi_0)} d\xi = \pi^{n/2} \).

Likewise \( \frac{1}{2} \frac{d}{dx} \left( f(x) \hat{f}(\xi) \right) = \lambda_k \int e^{-i(\xi \cdot \xi_0)} d\xi = \frac{1}{2} \lambda_k \int e^{i(\xi \cdot \xi_0)} d\xi = \pi^{n/2} \lambda_k \).

And \( \frac{1}{2} \frac{d}{d\xi} \left( f(x) \hat{f}(\xi) \right) = \left( \sum_{k=1}^{n} ||x_k - \xi_k|| \right)^{1/2} \frac{d}{d\xi} \left( f(x) \hat{f}(\xi) \right) = 2 \left( \sum_{k=1}^{n} ||x_k - \xi_k|| \right)^{1/2} \lambda_k \).

Thus \( \frac{1}{2} \frac{d}{dx} \left( f(x) \hat{f}(\xi) \right) = \lambda_k \int e^{-i(\xi \cdot \xi_0)} d\xi = \lambda_k \int e^{i(\xi \cdot \xi_0)} d\xi = 2 \pi^{n/2} \lambda_k \).

Next, we express another version of the uncertainty principle in terms of \( L^p \) norms.

This will also allow us to introduce a few more basic concepts about distributions.
Define: Let \( u \in S'(\mathbb{R}^n) \) be a tempered distribution. We define \( \text{supp}(u) \) to be the smallest closed subset of \( \mathbb{R}^n \) such that \( u(x) = 0 \) for all \( f \in S(\mathbb{R}^n) \) with \( \text{supp}(u) \subseteq \text{supp}(f) \). We say a distribution \( u \in S'(\mathbb{R}^n) \) is of compact support if \( \text{supp}(u) \) is bounded. We define \( \mathcal{E}'(\mathbb{R}^n) \) to be all distributions of compact support.

Lemma: Let \( \mu \in S'(\mathbb{R}^n) \) with compact support. Then \( \mu \) has a natural extension to a map \( \mu : C^0(\mathbb{R}^n) \to \mathbb{C} \) which is continuous with respect to the seminorms \( \| u \|_{\mu} = \sup_{\phi \in \text{supp}(\mu)} |\phi(u)| \).

Prop: Let \( u \in C^0(\mathbb{R}^n) \) with \( \text{supp}(u) \subseteq U \). Then for \( f \in S(\mathbb{R}^n) \) we have \( u_\mu(x) = u(x) \cdot 1_{\text{supp}(\mu)} \). Thus \( \| u \|_{\mu} = \sup_{\phi \in \text{supp}(\mu)} |\phi(u(x))| \).

Thus, \( \| u \|_{\mu} \leq \| \mu \|_{\text{supp}(\mu)} \| u \|_{C^0(\mathbb{R}^n)} \). Now this makes sense even for \( f \in C^0(\mathbb{R}^n) \).

So the extension property on continuity with respect to the seminorms follows at once.

The Fourier transform of compactly supported distributions is easy to describe "classically":

Prop: If \( u \in \mathcal{E}'(\mathbb{R}^n) \) then \( \hat{u} \in C^0(\mathbb{R}^n) \) and is given by the formula \( \hat{u}(\xi) = \hat{u}(\epsilon^i \cdot \xi) \) and satisfies an estimate \( \| \hat{u} \|_{L^1(\mathbb{R}^n)} \leq C \| u \|_{C^0(\mathbb{R}^n)} \) for some fixed \( C \). On the other hand,\n
if \( u \in \mathcal{E}'(\mathbb{R}^n) \) then \( u \in C^0(\mathbb{R}^n) \), is given by \( u(x) = \int_{\mathbb{R}^n} \hat{u}(\epsilon^i \cdot \xi) e^{ix \cdot \xi} \), and satisfies an estimate of the form \( \| u \|_{L^1(\mathbb{R}^n)} \leq C \| u \|_{C^0(\mathbb{R}^n)} \) for some fixed \( C \).

Prop: First define the function \( \hat{u}(\xi) = u(\epsilon^i \cdot \xi) \) and satisfies an estimate \( \| \hat{u} \|_{L^1(\mathbb{R}^n)} \leq C \| u \|_{C^0(\mathbb{R}^n)} \) for some fixed \( C \). Since the difference quotients

\[ \hat{A}_{\epsilon^i \cdot \xi} = \frac{1}{\epsilon^i} e^{i \cdot \xi} \text{ for } \xi \rightarrow 0 \]

in \( C^0(\mathbb{R}^n) \) are uniformly bounded on compact sets.
we get \( u_{n,k}(x) = u_k(\beta_{n,k}^{-1} x) \) \( \rightarrow u(x(1 - i\xi \xi)^{i\xi}) \). Induction gives \( \mathcal{F}_{\partial} u \) exists

and \( \mathcal{F}_{\partial} u(x) = \left( e^{2\pi i x \xi} \right)^{i\xi} \), thus \( \| \mathcal{F}_{\partial} u(x) \|_{C^0(x)} \leq C \sum_{n,k \in \mathbb{N}} \| \mathcal{F}_{\partial} u(x) \|_{C^0(x)} \) some compact \( K \subset M \). Thus \( \| \mathcal{F}_{\partial} u(x) \|_{C^0(x)} \) follows. Finally, to show \( \mathcal{F}_{\partial} u(x) \) is the FT of \( u \)

in the sense of distributions we need to show \( \langle u, \mathcal{F}_{\partial} u(x) \rangle \). Finally, to show \( \mathcal{F}_{\partial} u(x) \) is the FT of \( u \)

for all \( f \in C^0(M) \). By density, if continuity of FT, it suffices to consider \( f \in C^0(M) \).

Then the result follows from linearity and Riemann integration because for any sequence \( n \) of vertices \( \{ u_k(x) \} \rightarrow 0 \), we have \( \left| e^{2\pi i x \xi} \right| \left( \sum_{n,k \in \mathbb{N}} \| \mathcal{F}_{\partial} u(x) \|_{C^0(x)} \right) \)

and by smoothness of \( \{ u_k(x) \} \) we have \( \left| e^{2\pi i x \xi} \right| \left( \sum_{n,k \in \mathbb{N}} \| \mathcal{F}_{\partial} u(x) \|_{C^0(x)} \right) \)

as well.

Corollary: Let \( u \in L^1(M) \), then \( \exists \) a sequence \( u_n \in C^0(M) \) such that \( u_n \rightarrow u \)

(weakly in sense of distributions). Moreover, the convergence is uniform on \( M \) and \( u_n \)

satisfies \( \| u_n \|_{C^0(M)} \leq \| u_n \|_{L^1(M)} \). For \( C, M \), uniform in \( x, x_0 \).

Proof: Let \( u \in C^0(M) \) with \( u \geq 0 \), \( \forall x \in M \). Define \( u_n(x) = (x \cdot e^x) u_n(x) \)

where \( \widehat{x} = x \cdot e^x \), i.e. \( \widehat{x} = (x \cdot e^x) \). Since \( \widehat{x} \in \mathbb{R} \), we have \( \widehat{x} \in C^0(M) \), so \( \{ x \cdot e^x u_n(x) \} \in C^0(M) \). Fix \( f \in L^1(M) \),

then \( \langle u_n, f \rangle = \langle x \cdot e^x u_n, f \rangle = \langle u_n, (x \cdot e^x) f \rangle \). Thus, to show \( u_n \rightarrow u \) uniformly in our sense

we need \( u_n \rightarrow u \) and \( \langle x \cdot e^x u_n, f \rangle \) uniform on \( M \). Both of these follow

at once because \( \langle x \cdot e^x u_n, f \rangle = \langle x \cdot e^x u_n, f \rangle \). And \( (x \cdot e^x - 1) u_n \) in \( S \),

where \( \widehat{x} \in S(M) \) so \( \langle x \cdot e^x u_n, f \rangle \) uniform on \( M \). Both of these follow

in our sense.

We now use these regularizations to set up the basic convolution identities

For distributions.
**Definition:** Let \( u \in S'(\mathbb{R}^d) \) and \( f \in S(\mathbb{R}^d) \). Then we define \( u*f \) to be the function
\[
x \mapsto \langle u(x), f(x) \rangle.
\]

**Theorem:** Let \( u \in S'(\mathbb{R}^d) \) and \( f \in S(\mathbb{R}^d) \). Then:

1. \( u*f \in C^\infty(\mathbb{R}^d) \), and there exists an \( N > 0 \) depending only on \( u \) and \( \xi_0 \) depending on both \( u \) and \( f \) such that \( |\partial^k(u*f)(x)| \leq C \langle 1 + |x| \rangle^N \). One has \( \partial^k(u*f) = u*f*\partial^k f \).
2. One has \( (u*f)*g = u*(f*g) \).
3. \( \hat{u*f} = \hat{u} \hat{f} \) and \( \hat{(u*f)\cdot v} = \hat{u} \hat{f} \cdot \hat{v} \) when multiplication is defined by \( \langle f* u, g \rangle = \langle u, f*g \rangle \).
4. \( \hat{u*f} = \frac{1}{i \pi} \int_{\mathbb{R}^d} \hat{u}(y) e^{-iyx} \, dy \) and \( \hat{(u*f)\cdot v} = \hat{u} \hat{f} \cdot \hat{v} \).

**Proof:** 1) To show \( u*f \in C^\infty(\mathbb{R}^d) \) and \( \partial^k(u*f) = u*f*\partial^k f \) we use induction and the fact that the difference quotients in \( x \) for \( x \) fixed, \( \Delta_y f(x,y) = \frac{1}{i \pi} \langle f(x+y), -\delta(y) \rangle \), converge to \( \partial_x f(x,y) \) in \( S(\mathbb{R}^d) \).

To get the bound note \( \| u*f \|_{L^1} \leq C \| u \|_{L^1} \| f \|_{L^1} \) and \( \| u*f \|_{L^1} \leq C \| u \|_{L^1} \| f \|_{L^1} \). Therefore, \( \| \partial^k u*f \|_{L^1} \leq C \| u \|_{L^1} \| f \|_{L^1} \). To show the identities 2)-4) we use the same idea repeatedly. If \( u, v \in S(\mathbb{R}^d) \)
and \( u \rightarrow v \), then for \( f \in S(\mathbb{R}^d) \) \( u-f \rightarrow v-f \). This is a simple consequence of the formula \( \int (u, f \cdot y) \, dy = \int (u, f \cdot y) \, dy \) for \( g \in C^\infty_0(\mathbb{R}^d) \) which can be shown using the Riemann sum idea of the FT proof for \( L^1(\mathbb{R}^d) \) above. By density, this gives \( \langle u*f, g \rangle = \langle u, f*g \rangle \) for all \( u \in S(\mathbb{R}^d) \) and \( f \in S(\mathbb{R}^d) \) and \( u \rightarrow f \) follows.

Now 2)-4) all follow from their counterparts in the case \( u \in S(\mathbb{R}^d) \) as well.

Now we return to uncertainty principles for the Fourier transform. The following estimate is often useful in the theory of PDEs:
Bernstein's Inequality: Let $\mu \in S'(\mathbb{R}^n)$ be a tempered distribution such that

$\hat{\mu}$ is supported in a rectangle $R = \{ x \in \mathbb{R}^n | |x_k| \leq \lambda_k^3 \}$ for some $\lambda_k \in \mathbb{R}^n$ and $\lambda_k > 0$. Then if $\mu \in S'(\mathbb{R}^n)$ one has $\mu \in S'_{\mathbb{R}^n}$ and there is a fixed $C > 0$ (not depending on $\mu$) such that:

$$\|\mu\|_{C_{\mathbb{R}^n}} \leq C \|\mu\|_p$$

when $|R| = \int_R|dx|$, where $\mu \in S'(\mathbb{R}^n)$ is the measure of $R$.

Proof: By multiplying $\mu$ by $e^{i\lambda_1 x}$ we can assume $\lambda_1 = 0$. Let $\mu \in C^\infty_0(\mathbb{R}^n)$ be

a function with $\mu = 1$ on the box $|x_k| \leq 2$ for $k = \ldots, n$, and set $\mu_R(x) = \mu(A^{-1}x)$

where $A = \det (a_{ij}), \ldots, n)$. Then $\mu \cdot \mu_R = \mu_R$, so by the Fourier inversion formula we

have $\mu = \mu_R(\hat{\mu})$. By interpolation we only need to show both $\|\mu\|_{C_{\mathbb{R}^n}} \leq C$ and $\|\mu\|_p \leq C \|\mu\|_{C_{\mathbb{R}^n}}$.

The second bound follows at once from $\|\mu_R(\hat{\mu})\|_p \leq \|\mu\|_{C_{\mathbb{R}^n}}$ while the second

follows from the formula $\|\mu_R(\hat{\mu})\|_p = |A| \|\mu\|_{C_{\mathbb{R}^n}}$ and $\mu \in S'(\mathbb{R}^n)$. 
