I. LCH Spaces and Radon Measures

A. Basics of LCH spaces

Lemma: Let X be a Hausdorff space and set $X' = \{x, y\} \setminus \{x, y\}$, and

$F(X) = \{F \in X : F \text{ is compact}\}$. Then $F(X)$ is a topology on $X'$ making it into a compact space. Finally, $i: X \to X'$ is an embedding.

Proof: First note that if $F \in X'$ then $F \subseteq \{x, y\}$, and $\{x, y\}$ is compact in $X$. In addition, $F(X') = F(X)$ which is open in $X$.

Thus, we only need to show that $U, V \in X'$ are compact in $X$ implies $U \cup V, U \cap V \in X'$, which is compact in $X$, and $U(X') = X' \setminus (U \cup V)$ is compact in $X$. Then $U \cup V$ is compact in $X$. These are all obvious.

Finally, let $X' = F \setminus \{x, y\}$. Then some $V_{x} = X' \setminus \{y\}$ because $x \in V_{x}$. Then $V_{x} \subseteq \{x, y\}$, and $V_{x}$ is always open. Thus $\{x, y\} \subseteq \{x, y\}$ and $V_{x}$ covers $X'$.

Finally, $i: X' \to X'$ is open all $V \subseteq X$. Also $i(U)(X')$ is open all $V \subseteq X$.

(LCH)

Defn: A Hausdorff space is called "locally compact" if $X$ is Hausdorff, we call $X'$ the "one-point compactification" of $X$.

Lemma: If $X$ is a LCH and $C \subseteq X$ with $C$ compact and $U$ open. Then $f: C(X, \mathbb{R})$

with $f|C = 1$ and $f = 0$ outside some compact subset of $U$.

Proof: Note that $C \subseteq X'$ as compact map open. Thus $X' \setminus U$ is closed and thus exists $C \subseteq X'$ with $C \subseteq X'$ and $U = X'$.

Thus $C \subseteq X'$, $C \subseteq X'$, and $C \subseteq X'$ is compact in $X$. By Urysohn's Lemma, thus exists

$f: C(X', \mathbb{R})$ with $f|C = 1$ and $f|C = 0$. Thus supp$(f) = \{x \in x' \mid f(x) \neq 0\} \subseteq C \subseteq X$ is compact.
Restricting $f$ to $X$ yields the desired function.

Remark: Note that one of the steps in the proof shows if $E \subseteq X$ is compact and $U$ open then
3. open $V$ with $E \subseteq V \subseteq U$ with $V$ compact.

Defn: Let $X$ be a topological space. If $U \subseteq X$ is an open set then we define $f \in C(U)$
we write $f^{-1}(U)$ and $\text{supp}(f)$ is a compact subset of $U$. If $E \subseteq X$ is a compact
subset we write $E \subseteq E^{-1}(U)$, and $\text{supp}(f)$ is compact. Thus, the conclusion of the
previous result can be written as $E \subseteq E^{-1}(U)$.

Lemma: Let $X$ be a LCH space and $E \subseteq \bigcup_{i=1}^{n} U_{i}$ a compact subset with open covering $U_{i}$.
Then there exists a collection of $V_{i} \subseteq C(X)$ with $V_{i} \subseteq U_{i}$ and $E \subseteq \bigcup_{i=1}^{n} V_{i}$.

p5: As shown in the Remark above, for each $x \in E$ with $x \notin U_{i}$, 3. an open set $V_{i}$ with
$x \not\subseteq V_{i}$ and $\overline{V}_{i}$ connect. Let $V_{2}, V_{3}, \ldots, V_{m}$ cover $E$. For each $i$ set
$F_{i} = \bigcup_{j \neq i} V_{j}$. Then $F_{i} \subseteq U_{i}$ are compact and $E \subseteq \bigcup_{i=1}^{n} F_{i}$. Now let $F_{i} \subseteq C(X)_{(0,1)}$
with $F_{i} \subseteq \bigcup_{i=1}^{n} U_{i}$. Then $\frac{1}{n} F_{i} \subseteq F_{i}$ on $C$ and $\frac{1}{n} F_{i} \subseteq \bigcup_{i=1}^{n} \overline{V}_{i}$. Let $\psi \in C(X)_{(0,1)}$
be given so that $C_{\psi}(X) \subseteq \bigcup_{i=1}^{n} \overline{V}_{i}$. Setting $U_{i} = \frac{1}{n} F_{i} \subseteq \bigcup_{i=1}^{n} \overline{V}_{i}$ gives the desired result.

B. Radon Measures

Defn: Let $X$ be a LCH space. Then a "Radon measure" on $(X, B_{X})$ is a Borel measure
such that $\mu(E) < \infty$ for a compact $E \subseteq X$, and such that:
1) (Outer Regularity) $\mu(E) = \inf \{\mu(U) : U \supseteq E \text{ and } U \text{ open}\}$ all $E \subseteq B_{X}$,
2) (Inner Regularity on open sets) $\mu(U) = \sup \{\mu(E) : E \subseteq U, E \text{ compact}\}$ all $U \text{ open}$.
If $F: \mathbb{R} \to \mathbb{R}$ is monotone increasing and right continuous there is a unique Radon measure on $\mathbb{R}$ such that $\mu((a,b)) = F(b) - F(a)$. And all Radon measures on $\mathbb{R}$ are given in this way. In fact one only needs to assume $\mu(e) < \infty$ all $e \in \mathbb{R}$ compact and we even get $\mu$ a inner regular on all Borel sets (get back to this in general later).

**Definition:** Let $X$ be a LCH space. We set $C_c(X)$ the set of all $f \in C(X)$ such that $\text{supp}(f) = \{x \in X : f(x) \neq 0\}$ is compact. A "positive linear functional" on $C_c(X)$ is a linear map $I: C_c(X) \to \mathbb{R}$ such that $I(1) > 0$ all $B \in \mathbb{R}$.

**Proposition:** Let $X$ be a LCH space and $I$ a positive linear functional on $C_c(X)$. Define the function $\mu: X \to \mathbb{R}$ by $\mu(U) = \sup\{I(B) : B \subseteq U\}$, and $\mu(\emptyset) = 0$. Then the function $\mu^* : P(X) \to \mathbb{R}$ defined by $\mu^*(A) = \inf\{\mu(U) \mid A \subseteq U \text{ open}\}$ is an outer measure.

**Proof:** By a U.C.A result we just need to show if $U = \bigcup_{i=1}^{\infty} U_i$ then $\mu^*(U) \leq \sum_{i=1}^{\infty} \mu(U_i)$, because then $\mu^*(A) = \inf\{\sum_{i=1}^{\infty} \mu(U_i) \mid A \subseteq \bigcup_{i=1}^{\infty} U_i \text{ is an open cover}\}$.

To show the inequality let $f : Y \to U$ and set $K = \supp(f)$. Then $K \subseteq \bigcup_{i=1}^{\infty} U_i$ so let $U_i \in C(X)$ be given with $U_i \subseteq Y_i$ and $K \subseteq \bigcup_{i=1}^{\infty} U_i$. Then $f = \sum_{i=1}^{\infty} f_i \chi_{U_i}$ and since $f_i \chi_{Y_i}$ we get $I(f) = \sum_{i=1}^{\infty} I(f_i) \leq \sum_{i=1}^{\infty} \mu(U_i) \leq \sum_{i=1}^{\infty} \mu(U_i)$. Taking the sup over all $f : Y \to U$ gives the result.

**Proposition:** Let $X, I, \mu$ be as above. Then all open sets are $\mu^*$-measurable.

In particular $\mu^*$ induces a Borel measure $\mu$ on $X$ with $\mu(U) = \sup\{I(B) : B \subseteq U \text{ open}\}$
we need to show that for all sets $A \in X$ with $\mu(A) < \infty$ that

$\mu(A \cap U) + \mu(A \cap V) \leq \mu(A)$ for all $U, V$ open. Given $\varepsilon > 0$ choose $V$ open

with $\mu(V) < \mu(A) - \epsilon$. Then $\mu(A \cap V) = \mu(A \cap V) + \mu(V)$ we get

$\mu(A \cap U) + \mu(A \cap V) \leq \mu(A)$ for all $\varepsilon > 0$. So it suffices to prove the inequality for

open sets of finite outer measure. Let $V$ be such a set.

For $\varepsilon > 0$ choose $F \in U(V)$ with $\mu(F) > \mu(U) - \varepsilon$. Since $V \supseteq U(F)$

is open and finite measure we can also find $g \in U \supseteq U(F)$ with $\mu(g) > \mu(U \supseteq U(F)) - \varepsilon$.

Then $f + g \in F$ so $\mu(f) = \mu(f) + \mu(g) - \mu(g) = \mu(U) - \mu(F) - \varepsilon$. Take $\varepsilon = 0$.

Proposition: Let $X, I, \mu$ be as above. Then $\mu$ is a Radon measure, and in fact one has

$$\mu(E) = \inf \{ \mu(V) : E \subseteq V \}$$

for all $E \in X$ compact.

Proof: Fix $E$ compact and let $E \subseteq V$ open. Then $E \subseteq E \subseteq V$. Thus $\inf \{ \mu(V) : E \subseteq V \}$

$\leq \inf \{ \mu(V) : E \subseteq V \} \leq \mu(V)$. Since this works for all $E \subseteq V$ we certainly have

$\inf \{ \mu(V) : E \subseteq V \} \leq \mu(E)$. On the other hand let $E \supseteq V$ and set $U = \{ x \in X : f \in I(V) \}$

Then $E \subseteq V$ and for all $g \subseteq U$ we have $\mu(g) < \mu(V) - \varepsilon$ so $\mu(I(V)) > \frac{1}{2} \mu(V) - \varepsilon$.

We may assume such $g$ gives $\mu(E) \leq \mu(V) - \frac{1}{2} \mu(V)$, and taking $\varepsilon > 0$ gives $\mu(E) = \mu(V)$ all $E \subseteq V$.

Thus $\mu(E) = \inf \{ \mu(V) : E \subseteq V \}$ and we are done.

This main identity $(\star)$ also shows $\mu(E) \leq \mu(E)$ for all compact $E \subseteq X$.

Finally, if $V \subseteq X$ open choose $\delta < \mu(V)$ and $f \subseteq V$ with $\mu(f) > \delta$. Let $E = \sup \{ \mu(V) : E \subseteq V \}$

then for any $E \subseteq V$ we have $\mu(E) > \delta$ so $\mu(I(V)) > \delta$, and by the main identity of the

proposition we get $\mu(E) > \delta$ as well. Thus $\inf \{ \mu(V) : E \subseteq V \} > \mu(E)$, and

the other direction is immediate from monotonicity.
Theorem: Let $X$ be a LCH space and $I: C_c(X) \to \mathbb{C}$ a positive linear functional. Then there exists a unique Radon measure $\mu$ such that $I(f) = \int_X f \, d\mu$. Moreover, $\mu(U) = \sup \{ |I(f)| : f \in C_c(U) \}$ for all $U$ open, while $\mu(E) = \inf \{ |I(f)| : f \in C_c(E) \}$ for all $E$ compact.

Proof: We have already given the construction $I \mapsto \mu$ where $\mu$ is a Radon measure with the correct formulas for $\mu(U)$ and $\mu(E)$.

To get the integral formula, by splitting into real and imaginary parts, thus positive and negative parts, and finally rescaling, it suffices to show $\frac{1}{N} \sum_{j=1}^N \| \mu(K_j) \| \leq I(f)$ for all $f \in C_c(X; \ell_1)$. Given such an $f$, choose $N \in \mathbb{N}$ and for $j = 1, \ldots, N$ and define $K_j = \{ x \in X : |f(x)| \leq \frac{1}{N} \}$.

Since $K_j$ are closed and $\sup f(X) = \bigcup K_j$, they are compact. For each $N \in \mathbb{N}$ set

$$|f(x)| = \min \{ \max \{ |f(x)|, 0 \}, \frac{1}{N} \}.$$ 

Thus, $N f(x) \in K_j$ if $f(x) \leq \frac{1}{N}$, and if $f(x) > \frac{1}{N}$, $|f(x)| \leq \frac{1}{N}$. In addition, we have $\frac{1}{N} \sum_{j=1}^N \mu(K_j) \leq \frac{1}{N} \sum_{j=1}^N \mu(K_j)$ and we get

$$\frac{1}{N} \sum_{j=1}^N \mu(K_j) \leq N |f| \mu(U) \leq \frac{1}{N} \sum_{j=1}^N |f(K_j)|.$$

On the other hand if $K_j \in \mathcal{U}$ open we get $N f \in \mathcal{U}$ so $I(f) \leq N |f| \mu(U)$, and taking $N \to \infty$ over $K_j \in \mathcal{U}$ we get $I(f) \leq N |f| \mu(U)$, but since $K_j \in \mathcal{K}$, we also have $N |f| \mu(U) \leq I(f)$ by the previous proposition. Thus $\frac{1}{N} \sum_{j=1}^N \mu(K_j) \leq I(f)$ as well.

Taking the difference yields $I(f) - \frac{1}{N} \sum_{j=1}^N \mu(K_j) \leq \frac{1}{N} \sum_{j=1}^N \mu(K_j)$. Taking $N \to \infty$ gives the desired result.

Finally we prove the uniqueness result. If $\hat{\mu}$ is any Radon measure such that $I(f) = \int_X f \, d\mu$ for all $f \in C_c(X)$. If $U \in \mathcal{U}$ is open and $f \in C_c(U)$ then $0 \leq f \leq N$ so $f \in C_c(X)$.

Integration gives $I(f) = \hat{\mu}(U)$. On the other taking $K_j \in \mathcal{K}$ compact with $K_j \to \hat{\mu}(U)$, we can find $K_j \in \mathcal{K}$ so integration gives $\int_X f \, d\mu = \hat{\mu}(U)$. Thus

$$\hat{\mu}(U) = \sup \{ |I(f)| : f \in C_c(U) \} = \mu(U),$$

where $\mu$ is the Radon measure constructed above.

Since $\hat{\mu}(E) = \inf \{ |I(f)| : f \in C_c(E) \}$, all $E \in \mathcal{E}$ we are done.