I. Stationary Phase

Our goal now is to compute the composition $A \circ B$ of two operators $A \in \mathcal{O}(\mathbb{R}^n), B \in \mathcal{O}(\mathbb{R}^{2n})$.

This boils down to the evaluation of certain oscillatory integrals.

Then general form of the problem is to estimate $I(\lambda) = \int e^{i\lambda f(x)} u(x) dx$

where $u \in C^\infty_0 (\mathbb{R}^n)$.

A. Non-stationary Phase

We have already seen the behavior with FSD.

Lemma: Let $d\lambda \to 0$ in $\mathbb{R}$. Then for $u \in C^\infty_0 (\mathbb{R}^n)$ with $supp(\hat{u}) \subseteq B_1(0)$, given $N \geq 3$, $C(\mathbb{R}^{2n})$

such that $|I(\lambda)| \leq C(\mathbb{R}^{2n}) \lambda^N \int_{\mathbb{R}^n} |\hat{u}(\xi)| d\xi$ when $|\lambda| \to 0$.

Proof: Set $L = \frac{1}{1 + |\xi|^2} \hat{u} \in C^\infty_0 (\mathbb{R}^n)$. Then $\frac{1}{\lambda} \hat{L} - (\frac{1}{\lambda})' = e^{i\lambda \xi} \hat{u}$. So $I(\lambda) = \frac{1}{\lambda} \hat{L}^{-1} e^{i\lambda \xi} \hat{u} \mid_{\mathbb{R}^n}$

and the rest follows from $|I(\lambda)| \leq C(\mathbb{R}^{2n}) \lambda^N \int_{\mathbb{R}^n} |\hat{u}(\xi)| d\xi$.

B. Gaussian Stationary Phase

The problem with the previous argument is when $A$ some points $x \in \mathbb{R}^n$ where $d\lambda (x) = 0$.

A model example of this is when $\phi(x) = \frac{1}{2} \langle x, Ax \rangle$ where $Q \in \mathcal{S}(\mathbb{R}^n)$

is a real non-symmetric matrix. To understand the behavior

we need the following lemma:

Lemma: Let $\frac{\hat{\phi}(0)}{\hat{\phi}(0)} = e^{i\frac{1}{2} \langle x, Ax \rangle}$, where $Q \in \mathcal{S}(\mathbb{R}^n)$, $n \geq 1$. Then $\widehat{\phi}$ (in the sense of tempered distributions) is given by the formula $\widehat{\phi}(\xi) = \frac{1}{(2\pi)^{n/2}} e^{i\frac{1}{2} \langle x, Q^{-1} \xi \rangle}$.

The $\Sigma(\phi) = \# \text{ real eigenvalues } - \# \text{ complex eigenvalues of } Q$.

Proof: Note that $\frac{\hat{\phi}(\lambda x)}{\hat{\phi}(0)} = \hat{\phi}(\lambda x)$ for any rotation matrix $\sigma \in \mathcal{S}(\mathbb{R}^n)$.

In particular we have after a rotation $Q \phi \phi^{-1} = (a_1, \ldots, a_n)$ where $\sigma \phi = 0$. 


Thus for it suffices to consider the case of $Q$ a diagonal matrix. 

Next we compute the one dimensional case $\mathcal{F}_1(e^{i\lambda x})$. In general for $2\in\mathbb{R}$ with both $\text{Re}\lambda > 0$ and $\text{Re}\lambda < 0$ we get $\mathcal{F}_1(e^{i\lambda x}) = \frac{(2\pi)^n}{2} e^{i\lambda x} e^{-\frac{1}{2}g^2}$. For fixed $\lambda$ both sides are analytic in $2$ and for $\text{Re}>0$, $\frac{d}{d\lambda} \mathcal{F}_1(e^{i\lambda x})$ so choosing the natural return of $\lambda$ we get $\frac{d}{d\lambda} e^{i\lambda x} = \frac{d}{d\lambda} e^{i\lambda x}$. Taking the natural limit as $\text{Re}\lambda \to 0$ gives $\mathcal{F}_1(e^{i\lambda x}) = \lim_{\text{Re}\lambda \to 0} \frac{d}{d\lambda} e^{i\lambda x} e^{-\frac{1}{2}g^2}$.

Thus by taking a product of such results we have
\[
\mathcal{F}_1(e^{i\lambda x}) = \lim_{\text{Re}\lambda \to 0} \frac{d}{d\lambda} e^{i\lambda x} e^{-\frac{1}{2}g^2}.
\]

Using this we have the following important result:

**Definition:** Let $Q \in \mathbb{R}^{n \times n}$ and $Q \in \text{Sym}(n)$ be non-degenerate. Then for $X/l$ and $X > 1$

we have the uniform estimate:
\[
\mathcal{F}_1(e^{i\lambda x}) = \sum_{k=0}^{n} \frac{1}{n!} |\lambda|^k |\lambda|^k
\]

where $|\lambda| < C(n) < n^{-1} \frac{1}{n!} \mathcal{D}(\frac{1}{2} \langle 0, 0 \rangle)^\frac{N}{2} |\lambda| |\lambda|$. 

**Proof:** Recall that $\sum_{n=0}^{\infty} = \sum_{n=0}^{\infty} \frac{1}{n!}$. Thus gives
\[
\mathcal{F}_1(e^{i\lambda x}) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{D}(\frac{1}{2} \langle 0, 0 \rangle)^n |\lambda| |\lambda|.
\]

In addition we have $\mathcal{F}_1(e^{i\lambda x}) \lambda \langle 0, 0 \rangle = \langle 0, 0 \rangle (0, 0) \mathcal{F}_1(e^{i\lambda x})$. Finally, we have:
\[
\sum_{k=0}^{n} \frac{1}{k!} |\lambda|^k |\lambda|^k
\]

and $\sum_{k=0}^{n} \frac{1}{n!} |\lambda|^k |\lambda|^k$. 

**Claim:** This is one of the main receptors we look at: $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on $\mathbb{R}^2$. Then $\text{sgn} \theta = e^{-i\theta}$

and $\frac{1}{2} \langle (y, z), (y, z) \rangle = -y \eta$ and $\frac{1}{2} \langle (y, z), (y, z) \rangle = -y \eta$. 

Plugging this into the above formula, we derive:

\[
(\Psi_x) \sum_{k=0}^{n} e^{ik(a+bi)} d_d \lambda^{k} E_{k} = Z_{\Psi_x}^{n}\left(\left(\sum_{k=0}^{n} a_{k} + i b_{k}\right)\right)
\]

C. General Stationary Phase

Now, return to the general case that started the discussion, compute \(I_0 = \oint e^{i \Psi(x)} \, dx \), \(u \in C_0^2(\Omega)\) modulo \(\Psi^0\). We'll start with the special case where \(u(x)\) has an isolated stationary point \(x_0 \in \Omega\). That is \(u'(x_0) = 0\) if \(x = x_0\).

The basic idea is just to change variables \(x\) to get to the Gaussian case above.

Lemma: (Morse Lemma) Let \(u: \Omega \to \mathbb{R}\) and let \(\nabla u(x) = 0\) with \(\nabla^2 u(x_0) \neq 0\). (The \(u \in C^2\))

Then \(\exists \, x_0 \in \Omega\) and a smooth map \(H: V \to B_{x_0}\), \(B_{x_0} = \{|1 < 1\}^n\) such that \(H(x) = (y(x), \ldots, y_n(x))\), where \(n_0 = \#\) of positive eigenvalues of \(\nabla^2 u(x_0)\).

Recall that \(\Psi(x)\) then \(u''(x_0) \in C^0(T_0(\mathbb{R}^n, \mathbb{R}^n)) \to \mathbb{R}\) is orientation defined

for some \(x_0 \in \Omega\). Given \(u: \Omega \to \mathbb{R}\) and \(x_0 \in \Omega\), \(u''(x_0)\) is orientation defined and under change

of variables \(y = \Psi(x)\), it is \(u''(x_0)|_{y=0} = \det \frac{\partial y}{\partial x}\).

Remark: It suffices to consider the case \(y(x_0) = 0\), and after rotation, we have \(u''(x_0) = \text{diag}(\lambda_1, \ldots, \lambda_n)\).

A further rescaling of the variables by \(y^* = \frac{1}{\lambda_1} y^1 \times \cdots \times \frac{1}{\lambda_n} y_n\) makes \(u''(x)\) of the form \(\text{diag}(1, \ldots, 1)\).

Translating gives \(x_0 = 0\). Thus, we may assume to be working with

\[
\Psi(x) = \frac{1}{2} \sum_{k=1}^{n} \left(\frac{x_k}{\lambda_k}\right)^2 + O(\|x\|^3). \quad \text{At } x = 0, \Psi(x) = \frac{1}{2} \sum_{k=1}^{n} \left(\frac{x_k}{\lambda_k}\right)^2 dt = \frac{1}{2} \sum_{k=1}^{n} \lambda_k x_k^2.
\]
for some smooth matrix \( g(x) \) with \( g(z) = \left( \begin{array}{c} 1 \\ z \\ \end{array} \right) \). Our goal is to find
\[ f(x) \text{ with } \langle x, g(x) \rangle = \langle x_0, g(0) \rangle, \text{ and we hope for } A(x) = A(x_0) + A^{(1)}(x_0) \]

Then \( g(x) = A^{(0)}(x) + A^{(1)}(x_0) \) is what we need.

Let \( F : M(n, \mathbb{R}) \to \mathfrak{sym}(n) \) be given by \( A \mapsto A^t - g(0) \). Then \( D^2 F_{g(0)}(A) = (DA)^t g(0)^t g(0) + g(0)^t (DA)^t \),

which is surjective by choosing \( h \neq 0 \) in \( \mathfrak{sym}(n) \). Therefore, by the implicit

function theorem, there exists \( G: \mathfrak{sym}(n) \to M(n, \mathbb{R}) \) with \( G \circ G = Id \). Then
\[ A(x) = G(g(0)) \text{ and the trick} \]

This allows us to prove our main theorem:

**Theorem (Stationary Phase):** Let \( x \in \mathbb{C}^n \) with a isolated critical point at \( x = 0 \) with \( \Psi''(0) \)

non-degenerate. Then for \( u \in C^1(\mathbb{R}^n) \) the real differential operator \( \Delta_{x}^{(0)}(D_x) \) of order \( \leq 2k \),

and a constant \( C = C_2(s_{\Psi''}(0) N) \) such that:
\[
|e^{i\lambda y} u(x)| = \frac{1}{2\pi} \int_{\mathbb{R}^n} e^{i\lambda x \cdot \xi} \Lambda_{\xi} |\xi|^{s_{\Psi''}(0)}\xi^N \Lambda_{\xi^N} |\xi|^{s_{\Psi''}(0)} d\xi \leq C x^{-\lambda} |u(x)|.
\]

Hence \( A = \lambda^{|s_{\Psi''}(0)|} e^{i\frac{\lambda}{2} |s_{\Psi''}(0)|} |u|^{-\lambda} \).

**Remark:** Let \( \xi = x + iy \) diagonalize \( \Gamma \) as before, in particular \( \langle y, \xi \rangle = \langle y_0, \xi \rangle + \frac{1}{2} \langle y, \xi \rangle \)

where \( \langle y, \xi \rangle = D_{x_0} G \Psi''(0) \). Changing the variable of integration (locally) we have
\[
S e^{i\lambda y} u(x) \, dx = \int e^{i\lambda (x \cdot \xi + |\xi|^2)} \frac{dy}{|\xi|^{s_{\Psi''}(0)}} \frac{dy}{|\xi|^{s_{\Psi''}(0)}}
\]
\[
= \int e^{i\lambda (y \cdot \xi)} \frac{dy}{|\xi|^{s_{\Psi''}(0)}} \, dy. \text{ Note that } |D_{x_0} G \Psi''(0)| = |D_{x_0} G| = |\Psi''(0)| \frac{1}{2}.
\]

This \( \langle \frac{\lambda}{2} \rangle \frac{1}{2} \langle \frac{1}{2} \rangle \) follows from Gaussian stationary phase.

**Example:** Here is the standard example of stationary phase. Let \( u \in C^2(\mathbb{R}^n \setminus 0) \).

and
\[
\phi(x) = \int e^{i\lambda x \cdot \xi} u(x) \, dx. \text{ Using polar coordinates this becomes}
\]
\[ f(x) = \sum_{n=0}^{\infty} e^{in(x+it)} \sum_{n=0}^{\infty} e^{in} \sin(n) \, dx \]

To figure out what's going on...

we can rewrite it as
\[ \sum_{n=0}^{\infty} e^{in(x+it)} \sum_{n=0}^{\infty} e^{in} \sin(n) \, dx = \sum_{n=0}^{\infty} e^{in} \sum_{n=0}^{\infty} e^{in(x+it)} \sin(n) \, dx \]

In local coordinates, let \( x = (x', x'') \) be the phase $y(x', x'') = x' \cdot x'' + x''^2$.

and we have
\[ dy = \left( 2x'' - \frac{2x'}{x''} \right) \, dx' \]

which makes equality when $x'' = x'$.

After a rotation, we can assume that is when $x'' = x' = 0$. Then $y''(0, 0) = I_0$.

Thus, by stationarity, since we have $|S_{xy}(y)\sum_{n=0}^{\infty} e^{in} \sin(n) | \leq \| x \|_2 \sum_{n=0}^{\infty} e^{in} \sin(n) \in C_0^{\infty}(\mathbb{R})$.

Thus $|\| y''(0, 0) \|_2 \leq C_0 \| x \|_2$. On the other hand, in the range $t \geq 2t+1$, we have

$|xw+1| = \| y \|_2$, so integration by parts with $g \in \mathcal{S}$ and $h \in \mathcal{S}$. 