I. Asymptotic expansions of symbols

Definition: Let \( a, b \in S^m \) be \( m \geq 1 \), and \( a \in S^m \). Then we say \( a \sim b \) if

\[
\forall k \geq 0 \quad a_k - b_k \in S^{m_k},
\]

This means that \( a \sim b \). Note that it is certainly possible for \( a \sim b \) to hold while \( a \neq b \). Likewise, one can have \( a \sim b \) with \( a \neq b \) for all \( j \). But in either case there is a unique modulo \( S^m \) in the sense that \( a \sim b \) holds in the first case, and \( a \sim b \) holds in the second.

Since asymptotic sums are purely formal, in some sense they always exist.

Proposition: Let \( m_j \to \infty \) and take a sequence \( a_j \in S^m \). Then there exists \( a \in S^m \), unique up to \( S^m \), such that \( a \sim a_j \).

Proof: Note that \( b \sim a \) as well, \( a - b = (a_k - b_k) - (b_k - b_k) \in S^m_k \) for all \( k \).

Now let \( a_j \) be given. The idea is to modify these to \( a_j \) where \( a_j - a_j \in S^m \). To do this, let \( \{a_j, b_j, c_j, \ldots\} \) be a countable collection of sequences defining \( S^m \), and choose \( b_j \in S^m \) with

\[
\|a_j - b_j\| < 2^{-j} \text{ for all } j, k \geq 1. \text{ We have for } \forall k \in \mathbb{N}, \text{ the bound}
\]

\[
\|a_k - b_k\| \leq \|a_k - c_k\| + \|c_k - b_k\| < 2^{-k} + 2^{-k+1}.
\]

Finally, we give some easier to check conditions which guarantee when an asymptotic sum exists. The upshot is that one only needs "decrease quickly" + "1o bounds".
Proposition: Let $\alpha, \beta \in C_{\kappa}(\mathbb{R}^+ \times \mathbb{R}^+)$ where $\alpha, \beta \to 0$. Suppose

1) $\int_{0}^{\infty} \left| \frac{d^k}{dt^k} \alpha(t) \right| dt < \infty$ for compact sets, and $M, N$ are some constants.

2) $\int_{0}^{\infty} \left| \frac{d^k}{dt^k} \beta(t) \right| dt < \infty$ for similar compact sets, and $M, N \to \infty$ (where $I$ is some interval $[0, \infty)$).

Then $\alpha \approx \beta$.

Proof: Let $k \in \mathbb{Z}$, which is guaranteed by the previous Proposition. We just need to show $\alpha \approx \beta$. By 1) and 2) we have both:

1) $\int_{0}^{\infty} \left| \frac{d^k}{dt^k} \alpha(t) \right| dt < \infty$ for all $x \in \mathbb{R}$, and

2) $\int_{0}^{\infty} \left| \frac{d^k}{dt^k} \beta(t) \right| dt < \infty$ for all $N$.

To interpolate between these bounds we need a lemma.

Lemma: Let $k \in \mathbb{Z}$, then $\exists$ compact $B$ with $k \in B$ and $C(B, \mathbb{R})$ such that for $\delta \in C(B, \mathbb{R})$

one has $\| \delta \|_{C(B, \mathbb{R})} \leq C \left( \| \frac{d^k}{dt^k} \delta(t) \|_{C(B, \mathbb{R})} \right)$.

Proof: This means to $\delta \in C(B, \mathbb{R})$, choose $\epsilon = \delta^k \| \frac{d^k}{dt^k} \delta(t) \|_{C(B, \mathbb{R})}$ to get $\| \delta \|_{C(B, \mathbb{R})} \leq C \| \frac{d^k}{dt^k} \delta(t) \|_{C(B, \mathbb{R})}$.

Taylor's theorem with remainder may be written as $\delta(x) = \delta^k(x) + \frac{\delta^{k+1}(x)}{k+1}$, some $x \in B$.

Now, three or two cases: Choose $\epsilon = \delta \| \frac{d^k}{dt^k} \delta(t) \|_{C(B, \mathbb{R})}$ to get $\| \delta \|_{C(B, \mathbb{R})} \leq C \| \frac{d^k}{dt^k} \delta(t) \|_{C(B, \mathbb{R})}$.

On the other hand, choose $\epsilon = \delta \| \frac{d^k}{dt^k} \delta(t) \|_{C(B, \mathbb{R})}$ to get $\| \delta \|_{C(B, \mathbb{R})} \leq C \| \frac{d^k}{dt^k} \delta(t) \|_{C(B, \mathbb{R})}$.

Using the lemma, we have an upper bound set of the form $K \times B, (\alpha_0)$, uniformly in $\Theta$.

The estimate $\int_{0}^{\infty} \left| \frac{d^k}{dt^k} \alpha(t) \right| dt < \infty$ holds, repeating this argument we have $\int_{0}^{\infty} \left| \frac{d^k}{dt^k} \alpha(t) \right| dt < \infty$, uniformly in $\Theta$. Thus $\Delta \approx \alpha_0$ for $\alpha_0$, as desired.
II. Kohn-Nirenberg Quantization.

A. "Change of Symbol" to WKB Quantization.

Let $A \in \mathcal{O}(\Omega)$ be properly supported. Modulo $L^\infty$ we may assume that $A$ is properly supported as a function of $(x,\xi)$.

In fact modulo $L^\infty$ we may assume that $A$ is properly supported as close to $0$ as we like. This leads to the idea that $A \equiv 0$ modulo $L^\infty$ should only depend on the "jet" of $A$ at $(x,\xi)$ along $\xi$. The precise result is:

Theorem: Let $A \in \mathcal{O}(\Omega)$ be properly supported. Then there exists $\sigma_h(x,\xi) \in S^m(\mathbb{R}^{n+1})$ such that

$$ A(x,\xi) = \sigma_h(x,\xi) \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi \cdot \eta} \sigma_h(x,\eta) \eta_d\eta. $$

Thus $A \equiv 0$ modulo $L^\infty$.

Moreover, we have

$$ \sigma_h(x,\xi) \approx \frac{1}{(2\pi)^{n/2}} P \int_{\mathbb{R}^n} e^{-i\eta \cdot \xi} \sigma_h(x,\eta) \eta_d\eta. $$

(\text{IEQ}) Since we are only counting $\sigma_h(x,\xi)$ modulo $L^\infty$, it suffices to assume $A$ is properly supported in $(x,\xi)$ in the sense that $A = 0$ near $(x,\xi)$ is properly supported in the usual sense.

Recall that $A$ properly supported implies that $A$ satisfies so that $A: C^\infty(\Omega) \to C^\infty(\Omega)$.

Define

$$ \sigma_h(x,\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi \cdot \eta} \sigma_h(x,\eta) \eta_d\eta, $$

which converges after sufficiently many IBP in the $y$ variable. We write $\sigma_h(x,\xi) = b_1(x,\xi) + b_2(x,\xi)$

where $b_1$ corresponds to $0: 1 - (x(x'(\xi\cdot\eta)))$ and $a: x(x'(\xi\cdot\eta))$ when $\lambda = 1$.

and $x(\xi \cdot \eta) \in C^\infty(\mathbb{R}^n)$ with $x(\xi \cdot \eta) \to 0$ for the first integral the phase $y(\xi \cdot \eta)$ is non-negative if we may IBP with $L = i \frac{\partial}{\partial y(\xi \cdot \eta)}$. Then

$$ b_1(x,\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi \cdot \eta} \int_{[\mathcal{A}(\mathbb{R}^n) \times \mathbb{L}(\mathbb{R}^n)]} dy_d \eta. $$

Adding $\partial^2_d^\alpha$ derivatives to this gives an integral $C(1,\xi_1,\xi_2,\xi_3) = 1 \cdot (\partial_1 \xi_1)^2 (1 \cdot (\partial_1 \xi_1)^2 + 1 \cdot (\xi_1)^2))$ where $\xi_1 = \xi_1(\xi_1)^2$.
We now turn to estimating the absolutely convergent integral

\[ b_2(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\eta} \left( \int_{-\infty}^{\infty} e^{iy\xi} \phi(\xi-\eta) d\xi \right) d\eta. \]

We have already studied the integral, and stationary phase gives:

\[ b_2(x,y) \approx \frac{x}{\sqrt{2\pi}} \frac{1}{\sqrt{\det(\nabla^2 \phi(\cdot,y))}} \left( \int \phi(\cdot,y) \right) + O(e^{-N}) \text{ for } N \gg 1. \]

In addition, note that \( \frac{1}{\sqrt{\det(\nabla^2 \phi(\cdot,y))}} = \frac{1}{\sqrt{\det(\nabla^2 \phi(\cdot,y))}} \sum_{k \in \mathbb{Z}} \frac{1}{\sqrt{\det(\nabla^2 \phi(\cdot,y))}} \]

On the other hand, the remainder term \( \left| \frac{1}{\sqrt{\det(\nabla^2 \phi(\cdot,y))}} \int \phi(\cdot,y) \right| \leq C(\lambda, \nu, \sigma) \left\langle \xi \right\rangle^{-10} \]

where \( C(\lambda, \nu, \sigma) \) basically depends on \( \sigma \) semi-rows of \( \xi \). At any rate we get

\[ |b_2(x,y)| \lesssim C(\lambda, \nu, \sigma) \left\langle \xi \right\rangle^{-10}. \]

Thus, to show \( b_2(x,y) \approx O(e^{-N}) \) we only need to show \( b_2 \) has moderate growth \( \left| \frac{1}{\sqrt{\det(\nabla^2 \phi(\cdot,y))}} \int \phi(\cdot,y) \right| \lesssim C(\lambda, \nu, \sigma) \left\langle \xi \right\rangle^{-10} \).

From the integral for \( b_2 \) (no stationary phase required).

Finally, let \( A_1 = O(e^{-N}) \), where \( c_0(x,y) = c_0(x,y) \left( x \right) \), \( x \in \mathbb{R}^d \), with \( x \neq 1 \) in \( \mathbb{R}^d \).

Then the above argument shows \( c_0 \to 0 \) uniformly in \( x \), and \( A_2 \to c_0(x,y) \).

Thus, for \( \left( x, y \right) \in S(m) \) we have by DCT

\[ \int_{\mathbb{R}^d} \frac{1}{\sqrt{\det(\nabla^2 \phi(\cdot,y))}} \int_{\mathbb{R}^d} \phi(\cdot,y) \int_{\mathbb{R}^d} \phi(\cdot,y) d\eta d\xi \approx \frac{1}{\sqrt{\det(\nabla^2 \phi(\cdot,y))}} \int_{\mathbb{R}^d} \phi(\cdot,y) d\eta d\xi. \]

On the other hand, by interchanging absolutely convergent integrals and Fourier inversion we have

\[ \int_{\mathbb{R}^d} \phi(\cdot,y) d\eta d\xi = \int_{\mathbb{R}^d} \frac{1}{\sqrt{\det(\nabla^2 \phi(\cdot,y))}} \int_{\mathbb{R}^d} \phi(\cdot,y) d\eta d\xi. \]