I. Kohn-Nirenberg Calculus

Recall from the previous lecture we may write any (at least properly supported)

\[ A \in \mathcal{Op}(S^*) \text{ as } A = \frac{1}{(2\pi)^n} \int e^{2\pi i \langle \xi, h \rangle} A \xi \eta(d\xi d\eta) = \sigma_A(h, \eta) \text{. Hence } \sigma_A \in \mathcal{S}^m(S^* \times \mathbb{R}^n) \]

and has asymptotic series \[ \sigma_A(h) = \sum \frac{1}{n!} \left( \frac{\partial^n}{\partial h^n} \frac{\partial^n}{\partial \eta^n} A(h, \eta) \right) |_{h=0}. \]

\[ \sigma_A(h) = e^{2\pi i A(h)} \]

This gives us:

Proposition: Let \( \mathcal{L}^m_{\text{Comp}}(S^*) \) be all \( A \in \mathcal{S}^m(\mathbb{R}^n) \) with a properly supported. Then the map

\[ \sigma: \mathcal{L}^m_{\text{Comp}}(S^*) \to \mathcal{S}^m(\mathbb{R}^n) \]

is given by \( A \mapsto \sigma_A \), where \( A \) acts by an isomorphism \( L^m(\mathbb{R}^n) \to S^m(\mathbb{R}^n) \).

\( \sigma_A(h) = e^{2\pi i A(h)} \) is smooth. And \( \sigma_A = \sigma_B \) mod \( S^m(\mathbb{R}^n) \).

Thus, questions about the calculus of \( \mathcal{Op}(S^*(\mathbb{R}^n \times \mathbb{R}^n)) \) can be reduced to questions about \( \mathcal{Op}((S^*(\mathbb{R}^n \times \mathbb{R}^n))) \).

Theorem: Let \( A \in \mathcal{Op}(S^*) \), then \( A^* \in \mathcal{Op}(S^*) \) and \( \sigma_{A^*} = Z_{\alpha \beta} \frac{1}{\alpha! \beta!} \frac{\partial^\alpha}{\partial h^\alpha} \frac{\partial^\beta}{\partial \eta^\beta} \sigma_A(h, \eta) \)

\( \sigma_A(h) = e^{2\pi i A(h)} \) is smooth. And the result follows from the asymptotic formula for \( \sigma_{A^*} \)

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\[ \sigma_{A^*}(h) = e^{-2\pi i A(h)} \]

We have \( \sigma_A = i Z \sigma_{A^*} \). Also, from the formula above \( \sigma_{A^*} = -i Z \sigma_A \eta \). Hence \( \sigma_{A^*} = -\sigma_A - i \sigma_A \eta \).

Using the general formula gives \( \sigma_{A^*} = i Z \sigma_A \frac{1}{\alpha! \beta!} \frac{\partial^\alpha}{\partial h^\alpha} \frac{\partial^\beta}{\partial \eta^\beta} (\eta, \eta) = -\sigma_A - Z \sigma_A \eta \)

We now discuss composition of operators.
Then: let $A \in C^\infty(\Omega)$, $B \in L^\infty(\Omega)$, with at least one of them properly supported, then $A \circ B \in C^\infty(\Omega)$ and we have
$$\sigma_{\lambda A} \sim \sum_{j=1}^d \lambda^{-d} \sigma_{\lambda B}(\lambda) \tilde{\sigma}_{\lambda A}(\lambda).$$

WLOG we can assume $A = \int_0^1 e^{i(r_0,\xi)} \sigma_{\lambda B}(\lambda_y) d\lambda_y$, $\xi$ properly supported. Define
$$C(r,\xi) = e^{i(r,\xi)} A (b_{\lambda B}(\xi_y)) d\lambda.$$ We'll show $C \sim \sum_{j=1}^d \lambda^{-d} \sigma_{\lambda B}(\lambda) \tilde{\sigma}_{\lambda A}(\lambda)$. Also, we'll show
$$\text{if } A_\epsilon = \int_0^1 \int_0^\infty e^{i(r,\xi)} \sigma_{\lambda B}(\lambda)(\xi_y) d\lambda_y d\lambda, \text{ then } C_\epsilon \in C \text{ with bounds } 1 \leq C \leq C_\epsilon \leq C_\epsilon(\lambda, \xi).$$

Thus $D\epsilon$ gives $A \circ B = \int_0^1 \int_0^\infty e^{i(r,\xi)} C_\epsilon(r,\xi) d\lambda_y d\lambda$. The computation of $C(r,\xi)$ is like before
$$C(r,\xi) = \int_0^1 \int_0^\infty e^{i(r,\xi)} \sigma_{\lambda B}(\lambda_y) \tilde{\sigma}_{\lambda A}(\lambda) d\lambda_y d\lambda \sim \sum_{j=1}^d \lambda^{-d} \sigma_{\lambda B}(\lambda) \tilde{\sigma}_{\lambda A}(\lambda).$$

This leads us to the following definition.

**Defn:** on $S^m/\mathbb{Z} \times S^m/\mathbb{Z}$ we define the bilinear form $A \circ B$ via $A \circ B \sim \sum_{j=1}^d \lambda^{-d} \sigma_{\lambda B} \tilde{\sigma}_{\lambda A}.$

The previous result can be written as: let $\sigma : L^\infty(\Omega) \to S^m/\mathbb{Z}$, then $\sigma_{\lambda A} \circ \sigma_{\lambda B} = \sigma_{\lambda A} \circ \sigma_{\lambda B}$.

**II. The Parametrix of an elliptic operator**

* We now give a typical application of the previous theory in action.

**Defn:** A symbol $p(x,D) \in S^m(\mathbb{R}^n)$ is called "elliptic" if for $x \neq 0$ and $0 \leq m \leq n$ such that $|p(x,D)(x)| = |x|^m$ for all $|x|^m \neq 0$. An operator $P \in L^\infty(\Omega)$ is called elliptic if its symbol is an elliptic symbol.

**Thm:** Let $P \in C^\infty(\Omega)$ be elliptic then there exists an elliptic $Q \in C^\infty(\Omega)$, unique up to $L^\infty(\Omega)$,
such that $P \cdot Q = Q \cdot P = I \mod 1^{\infty}$.

Let $p(x,y)$ be the (elliptic) symbol of $P = p(x,y)$. It suffices to consider an elliptic symbol $g(x,y) \in S^{-1}(\mathbb{R}^n)$ such that $p \cdot g \sim g \cdot p \sim 1$.

First, notice that the ring $\mathbb{C}[x,y]$ in $(x(1 + \lambda_1 x_1 + \cdots + \lambda_n x_n))_{\lambda \in \mathbb{N}}$ for $\operatorname{Id}_n$ can be chosen to be smooth. Set $q_0 = 1 - x(\frac{1}{p(1)}),\text{ then } x(1 + \lambda_1 x_1 + \cdots)$ with $\lambda \in \mathbb{N}_1$. A short calculation shows $q_0 \in S^{-1}(\mathbb{R}^n)$. We have $p \cdot q_0 = 1 - q_0$ since $x \in S^{-1}(\mathbb{R}^n)$. This is $e^{-y} (1 + x^2 + x^4 + x^8 + \cdots)$.

Now define $q = q_0 \cdot (1 + x + x^2 + x^3 + \cdots)$. Then $p \cdot q \sim (1 + x + x^2 + x^3 + \cdots) \sim 1$.

Thus $\hat{P}(x,y) \cdot Q(y) = Q(y)$ $\mod 1^{\infty}$ (be any order) $\hat{Q}(y)$ to make it properly supported.

On the other hand, a similar construction gives $\hat{Q}(y) \cdot L^{-1}(\mathbb{R}^n)$ with $\hat{Q} \cdot P = \operatorname{Id} \mod 1^{\infty}(\mathbb{R}^n)$.

Then $\hat{Q} \cdot P \cdot Q = Q \mod 1^{\infty}$, so $\hat{Q} \cdot Q = Q \mod 1^{\infty}(\mathbb{R}^n)$.

Corollary: Let $P \in L^1(\mathbb{R}^n)$ be elliptic, and let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then $\operatorname{sing supp}(\hat{P}u) = \operatorname{sing supp}(u)$.

In particular if $P \in C^0(\mathbb{R}^n)$, then $u \in C^0(\mathbb{R}^n)$.

By: We already know $\operatorname{sing supp}(\hat{P}u) \leq \operatorname{sing supp}(u)$. On the other hand if $\hat{Q}$ is a properly supported.

homomorphism for $\hat{P}$ then $\operatorname{sing supp}(\hat{Q}u) = \operatorname{sing supp}(u)$.