I. $L^p$ estimates for PDO.

We now turn to quantitative estimates for PDO. The main result we have in this regard is the following:

**Theorem:** Let $a(x,p) \in S^0(\mathbb{R}^m)$ with $A^-a(x,p)$ compactly supported on $\mathbb{R}^m$ as a distribution kernel. Then for each $\epsilon > 0$ there exists a decomposition $A^-a(x,p)$ where $A^-\text{supp}(\chi) \subseteq (M^s_t, \text{supp}(\tau))$ with $\text{supp}(\chi) \subseteq \text{supp}(A^-) + \{a\} \cdot B_s(\chi)$.

The symbol map $a(x,p) \mapsto a(x,p)$ is continuous from $S^0 \rightarrow S^{-\infty}$, possibly with constants depending on $\text{supp}(A^-)$.

In particular, for $s \geq 0$ and $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^m)$ such that $\psi \text{supp}(\chi) \subseteq (M^s_t, \text{supp}(\tau))$ for all $\chi \in C_c^\infty(\mathbb{R}^m)$, then $A^-\text{supp}(\chi) \subseteq \mathbb{R}^m$ with a bound depending on $\delta$, and possibly $\text{supp}(A^-) \subseteq \mathbb{R}^m$. The core of the proof of this statement is contained in the following lemma:

**Lemma:** Let $a(x,p) \in S^0(\mathbb{R}^m)$ have the properties of the previous theorem. Then for $\epsilon > 0$ there exists $B_s(\chi) \subseteq (M^s_t, \text{supp}(\tau))$ (as a distribution kernel) such that $(M^s_t)A^- \text{supp}(\chi) = B_s(\chi) + K$ where $K \subseteq L^\infty(\mathbb{R}^m)$. Finally, $C^1(\mathbb{R}^m)$ bounds for $K$ depend on $S^0$ bounds for $a$.

and possibly the size of $\text{supp}(A^-) \subseteq \mathbb{R}^m$ as a distribution.

**Proof of Lemma:** Let $C = (M^s_t)A^-$. Then $C^{\text{def}} = C$ (acting on $S(\mathbb{R}^m)$), and $C \subseteq L^\infty(\mathbb{R}^m)$.

Thus $\sigma_c \in S^0$, and $\sigma_c \overline{\sigma_{c'}} \in S^0$. In particular $\rho_c(x)^2 \leq \delta^2$ for $1 \leq \delta$, and $\text{Im}(\rho_c(x)^2)$. Thus, we can construct $b_c \in S^0$ with $b_c \overline{\sigma_c} \overline{\sigma_{c'}}$ for $1 \leq \delta$. This gives for $B_b \equiv (\delta b_c(x))^2$ the identity $C = B_c^2 + c, c \in C, \text{supp}(\chi) \subseteq (M^s_t, \text{supp}(\tau))$. Where we may assume $B_b \equiv (\delta b_c(x))^2$.

outside a compact set on $\mathbb{R}^m$ ($b_b = m^s_t$ for $b_b$ large), so using Hörmander-Nirenberg quantization we directly get $\frac{1}{\text{vol}(\mathbb{R}^m)} \int \text{supp}(S^0 \overline{b_c(x)}) \delta(x) = (M^s_t)\delta(x)$ for all large $\delta$.

Note that $C_c^{\text{def}} = C$ (on $S(\mathbb{R}^m)$). Next, let $G_{\delta}$ be a properly supported parametrix for $B_b$.,
\[ \text{In particular, if } B > a \text{ such that } Q_\alpha = (M\phi)^{-1}(\text{stray}), \text{ then } C_\alpha = 0 \text{ for all } \alpha \in \{B_\alpha(M\phi) \times B_\alpha(M\phi)\}^c. \]

By induction, we may now set \[ B_k = B_{k-1} + q^{k-1} C_{k-1} \] where \( C_k \) is given by \( C_{k+1} = B_k C_k \).

We have already motivated \( C_k \) and its construction above. Assume \( \psi \in L^2(M) \) we get \[ B_k B_k = (B_{k-1} + q^{k-1} C_{k-1}) = B_{k-1} B_{k-1} + q^{k-1} C_{k-1} C_{k-1} + \frac{1}{4} C_{k-1} B_{k-1} + B_{k-1} \psi \]

\[ \psi \cdot B_k B_k = (B_{k-1} + q^{k-1} C_{k-1}) \psi = B_{k-1} \psi + q^{k-1} C_{k-1} \psi + \frac{1}{4} C_{k-1} B_{k-1} \psi + B_{k-1} \psi. \]

Now, construct an operator \( B_k B_k + q^{k-1} C_{k-1} \) and after on \( L^2(M) \) measures we can also

\[ \text{supp} (B_k B_k) \subseteq B_k(0) \times B_k(0). \]

In section of the proofs in the last few lectures also shows the map \( \alpha \mapsto \varphi_0 \) is continuous. From \( S_0 \rightarrow S_0^* \), possibly with constants depending on \( \text{supp}(A) \subseteq C_0 \). (In this way, things should only depend on \( S_0 \) seminorms of \( \alpha = \varphi_0 \), but we will get to this later).

In the end, we have \( K = C - B_k B_k \in L^2(M) \), with \( S_0 \) seminorms depending only on \( S_0 \) seminorms of \( \alpha \).

Now, we return to the proof of our main theorem.

Of sorts, \( \varphi_0 \) that \( \psi \in \mathcal{P}(\varphi_0) \), be an even positive bump with \( \mathcal{S} \psi \psi \varphi_0 \). Then \( \psi(\varphi_0) \psi \varphi_0 \) is such

But \( 0 \leq \varphi_0 \) and \( \psi(\varphi_0) \geq 0 \). Let \( Q_\alpha = \varphi_0 (\psi(\varphi_0) \psi) \). We use this to decompose \( A = A_{\varphi_0} + A_{\varphi_0} \), where \( A_{\varphi_0} = A_{\varphi_0} \psi \). Now \( I - A_{\varphi_0} \rightarrow 0 \) in \( L^2(M) \). But \( K : L^2(M) \rightarrow L^2(M) \)

and the size of its support. Thus, for \( \lambda \) large enough, and by choosing \( K \) so that

\[ (M+i\psi)^{-1} I - A_{\varphi_0} = B_k + K \] we get \( \| A_{\varphi_0} \|_{L^2(M) \rightarrow L^2(M)} \ll \). On the other hand,
\[ A_{\lambda} \in L^{1/2}(\mathbb{R}^4), \text{ and since } \mathcal{D}_{k}\lambda(x) = \lambda^k \mathcal{D}_{k}\lambda(y) \text{ we get } \sup_{\lambda} \mathcal{D}_{k}\lambda \leq \frac{1}{k!} \lambda^{k-1} \mathcal{D}_{k}\lambda. \]

Note that the \( L^{1/2} \) bounds for \( \lambda \) depend on the \( \lambda \) directly because \( \sigma_{\mathcal{D}_{k}\lambda} = \sigma_{\lambda}^{1/2} \).

The size of \( \lambda \) in turn is determined by \( L^{1/2}(\mathbb{R}^4) \) bounds for \( \lambda \) above.

We now apply these estimates to Sobolev spaces.

\[ \text{II. Bounds for local Sobolev spaces.} \]

\[ \text{Def: Let } u \in \mathbb{R}^n, \text{ we define } H^s_{\text{loc}}(\mathbb{R}^n) \text{ to be all } u \in C^k_c(\mathbb{R}^n) \text{ such that } u \in C^k(\mathbb{R}^n). \]

\[ \text{For all } u \in C^k(\mathbb{R}^n). \text{ Note that this can be made into a Frechet space by taking a sequence } u_j \]

\[ \text{with } u_j \sup_{\mathbb{R}^n} = 0. \text{ We can also set } H^s_{\text{comp}}(\mathbb{R}^n) = \mathcal{L}^s(\mathbb{R}^n) \cap H^s(\mathbb{R}^n). \]

The main result we have here is the following.

\[ \text{Theorem: Let } A \in L^1(\mathbb{R}^4) \text{ be properly supported. Then for each } s > 0, \text{ one has } A : H^s_{\text{loc}}(\mathbb{R}^4) \to H^{s'}_{\text{loc}}(\mathbb{R}^4). \]

\[ \text{Proof: Let } u \in C^k_c(\mathbb{R}^n), \text{ then we need to show } \| A u \|_{H^{s'}_{\text{loc}}(\mathbb{R}^4)} \leq C(s, k, \| A \|_{H^s(\mathbb{R}^4)}) \| u \|_{H^s_{\text{comp}}(\mathbb{R}^4)}. \]

Since \( A \) is properly supported, we can find \( \psi \in C^\infty(\mathbb{R}^4) \) such that \( A = A \psi \). Then it suffices

\[ \text{to show } \hat{A} : H^s(\mathbb{R}^4) \to H^{s'}(\mathbb{R}^4) \text{ for } \hat{A} = A \psi \text{ a compactly supported version of } A \text{ on } L^2(\mathbb{R}^4). \]

\[ \text{Let } A \text{ be a properly supported parametrix for } L^2(\mathbb{R}^4). \text{ Then } \| \lambda_{\delta} A \|_{H^{s'}(\mathbb{R}^4)} \leq C(s, \| A \|_{H^{s'}(\mathbb{R}^4)}) \]

so it suffices to show \( A = A \delta A_{\lambda_{\delta}} : L^2(\mathbb{R}^4) \to L^2(\mathbb{R}^4) \) is bounded \( L^2 \) and compactly supported.

so the previous Theorem applies.

\[ \text{III. Application to Local Solvability.} \]

We can use the previous machinery to show that in some sense elliptic equations

always have local solutions.
Theorem: Let $A(D) = L(A,m)$ be an elliptic differential operator with $0 \in \sigma_p(D)$. Then

For each $\chi \in \mathcal{D}$, $\gamma \in \mathcal{D}$ open and $\chi \approx \gamma$, such that for all $f \in D(\gamma)$, $\exists \gamma \in D(\gamma)$ with $Au = f$ in $\gamma$. If $f \in \mathcal{D}(\gamma)$, then $u \in \mathcal{D}(\gamma)$ where $m$ is the order of $A$.

Proof: Since $A$ is elliptic, so is $A^*$. We already know that for each compact $K \subset \gamma$ and some $C = C(K, \varepsilon, \delta)$ such that $\|u\|_{H^1(\gamma)} \leq C\|u\|_{H^1(\gamma)}$, all $u \in \mathcal{D}(\gamma)$, there is a properly supported parameter for $A^*$.

Next, recall the Sobolev inequality for $u \in \mathcal{H}^1(\gamma) \cap \mathcal{L}^1(\gamma)$ which is $\|u\|_{\mathcal{L}^1(\gamma)} \leq C\|u\|_{\mathcal{H}^1(\gamma)}$.

Thus, for $\varepsilon$ small enough we get $\exists n \in \mathcal{H}^1(\gamma)$ such that $\sup \mathcal{H}(\gamma)$, thus $\|n\|_{\mathcal{H}^1(\gamma)} = \|u\|_{\mathcal{H}^1(\gamma)}$.

Proposition: (Abstract solvability) Let $X \rightarrow Y$ be Banach spaces with $X$ reflexive, and let $A: X \rightarrow Y$ be continuous, let $\mathcal{H}^1(\gamma)$ be a linear subspace. Then given $f \in Y$, a sufficient condition that one can find $u \in X$ with $\langle Au, y \rangle = 0$, all $y \in \mathcal{H}^1(\gamma)$ is that $A \subset C$ with $\|u\|_{\mathcal{H}^1(\gamma)} \leq \|A\|_{\mathcal{H}^1(\gamma)}$

for all $y \in \mathcal{H}^1(\gamma)$.

Proof: Assume the estimate $\|u\|_{\mathcal{H}^1(\gamma)} \leq \|A\|_{\mathcal{H}^1(\gamma)}$, all $u \in \mathcal{H}^1(\gamma)$. Then $A|_{\mathcal{H}^1(\gamma)}$ is bounded, so we may define on $\text{Ran}(A|_{\mathcal{H}^1(\gamma)})$ the linear function given by $h(\gamma) = \langle \gamma, \nu \rangle$ with $\nu = A^* \gamma$. Then $\|\nu\|_{\mathcal{H}^1(\gamma)} \leq C\|\gamma\|_{\mathcal{H}^1(\gamma)}$.

So by the Hahn-Banach theorem $\exists$ continues extension $\mathcal{H}^1(\gamma)$ to all of $X^*$. Say $X \times X^*$

then exists $u \in X$ such that $\langle u, \gamma \rangle = \langle \gamma, \nu \rangle$, all $\nu \in X^*$, and for $\nu = A^* \gamma$, $\nu \in \mathcal{H}^1(\gamma)$ we get $\langle u, A^* \gamma \rangle = \langle \gamma, \nu \rangle$. Thus $\langle Au, y \rangle = 0$, all $y \in \mathcal{H}^1(\gamma)$ as desired.
Note that for our application, $A: H^{s_0}(\mathbb{M}) \rightarrow H^{s}(\mathbb{M})$ and $B: H^{s}(\mathbb{M}) \rightarrow H^{s_0}(\mathbb{M})$, possibly after extending coefficients in a uniformly bounded way. Given $\psi \in L^2(\mathbb{S}^1 \times \mathbb{S}^2, \nu)$, we see that $\nu$ is a small, and $\mathcal{P} \in D'(\mathbb{S}^1 \times \mathbb{S}^2)$.

We can find $\psi \in L^2(\mathbb{S}^1 \times \mathbb{S}^2)$ with $\psi = 1$ on $\mathcal{V}$, $\nu \in L^2(\mathbb{S}^1 \times \mathbb{S}^2)$.

Then we can set $\chi = H^{s_0}(\mathbb{M})$, $\psi = H^{s}(\mathbb{M})$, and $\nu = H^{s_0}(\mathbb{M}) \cap C^1(\mathbb{M})$ for $\mathcal{V}$ small.

We can set $\nu \in H^{s_0}(\mathbb{M})$ and $\chi \in H^{s}(\mathbb{M})$, $\nu \in C^1(\mathbb{M})$, $\nu \in L^2(\mathbb{M}) \cap C^1(\mathbb{M})$ for $\mathcal{V}$.

Then we find $\psi \in L^2(\mathbb{S}^1 \times \mathbb{S}^2)$ with $\psi = 1$ on $\mathcal{V}$, $\nu \in L^2(\mathbb{S}^1 \times \mathbb{S}^2)$.

Thus we end here with an interesting example of Hans Lewy.

Theorem: On $\mathbb{R}^3$ define the operator $P = \overline{\delta} + i \delta_y$, where $(x, y, z) \in \mathbb{R}^3$, $\overline{\delta} = \partial_\zeta (x, \partial_y \zeta)$, $\delta_y = \partial_{\zeta} (x, \partial_y \zeta)$, $z = x + iy$.

Then there exists $\mathcal{C} \in C^0(\mathbb{S}^3)$ such that there is no $\mathcal{O}$, and $u \in D'(\mathbb{S}^1 \times \mathbb{S}^2)$

with the property $Bu = P$ on $\mathcal{V}$.

**Note:** $P = \frac{1}{2} (i \delta_\zeta - \overline{\delta}) = (x + iy) \partial_z$, which is not elliptic.

**Proof:** First recall the polar form of vector fields: $\partial_x = \cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta$, $\partial_y = \sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta$.

Thus $\overline{\delta} = e^{i \theta} \partial_r + \frac{i e^{i \theta}}{r} \partial_\theta$. Let $u(x, y)$ be a function and set $\mathcal{V}(x, y) = \int_{\mathcal{V}} u \, dx$.

We then compute $\int_{\mathcal{V}} \partial^2 u \, dx = \int_{\mathcal{V}} \overline{\delta} u \, dx$, using $dx = i \partial_\theta$.

Then it is $\frac{C}{2} \int_{\mathcal{V}} e^{i \theta} \partial_r \partial_\theta \, dx = \frac{1}{2} \int_{\mathcal{V}} \partial_r \partial_\theta u \, dx = 2 \partial_z \int_{\mathcal{V}} \partial_x u \, dx = 2 \partial_z u$

on the other hand if $\overline{\delta} u = -i \partial_y u + \partial_x u$ we get $\partial_y \overline{\delta} u = -2 i \partial_y u + 2 \partial_z u$.

Thus $\overline{\delta} u = \frac{1}{2} \int_{\mathcal{V}} \partial_x u \, dx$, where $\partial_x = \frac{1}{2} (i \delta_\zeta - \overline{\delta})$, $\mathcal{V} = \int_{\mathcal{V}} u \, dx$, $\overline{\delta} u \overline{\delta} u = \mathcal{V}$.

Now let $P = \frac{1}{2} \int_{\mathcal{V}} u \, dx \in C^0(\mathbb{S}^3)$. Then $\frac{1}{2} \int_{\mathcal{V}} u \, dx \, dx = \pi (\partial_x u)$.

Now from the function $Q = i \mathcal{V} + \pi \mathcal{V}$, solves $\overline{\delta} Q = 0$, and is defined for $\mathcal{V}(\omega) = 0$, with this close to $\omega = 0$. In addition, $Q$ counts, is continuous, and $Q|_{\omega = 0}$ is real.
By Schwartz reflection $Q$ extends to a holomorphic function close to $w=0$.

In particular $Q|_{|z|=1}=\pi f(z)$ must be real analytic in a nod of $t=0$. But we could take $u=\chi \frac{z}{\sqrt{t}}$ close to $t=0$, which means that $u$ is real (at least $C^1$) solving $\partial_t u + (1+\mu) u = 0$. 