# MATH 110
# SOLUTIONS TO HW #7

## Section 6.1, problem 5.

Since the boundary conditions are rotationally symmetric, we will try to find a solution that is only a function of $r = \sqrt{x^2 + y^2}$. That is, we will try to find a function $u(r)$ that is a solution to the problem:

$$\begin{aligned} \partial_r^2 u + \frac{1}{r}\partial_r u &= 1 , && \text{in } 0 \leqslant r < a , \\ u(a) &= 0 . \end{aligned}$$

There are several ways to go about solving this. The first is to use general ODE techniques. That is, we first make the substitution $w = \partial_r u$ and try to find solutions to:

$$\partial_r w + \frac{1}{r} w = 1 . \tag{1}$$

The way to do this (via theory) is to first find all possible solutions:

$$\partial_r W + \frac{1}{r} W = 0 ,$$

with initial consitions $W(s) = 1$. We will call these functions $W(r; s)$. That is:

$$\begin{aligned} \partial_r W(r;s) + \frac{1}{r} W(r,s) &= 0 , \\ W(s;s) &= 1 . \end{aligned}$$

Then we will in general have that:

$$w(r) = \int_b^r W(r;s) ds .$$

This is the so called "variation of parameters formula" for the inhomogeneous equation (1).

A simple calculation (e.g. using an integrating factor) shows that we have $W(r;s) = \frac{s}{r}$, so we in fact have that the general solution to (1) is:

$$\begin{aligned} w(r) &= \int_b^r \frac{s}{r} ds , \\ &= \frac{1}{2} r + \frac{C_1}{r} , \end{aligned}$$

where $C_1$ is an appropriate constant. Next, from the substitution $w = \partial_r u$ we have that:

$$u(r) = \frac{1}{4} r^2 + C_1 \ln(r) + C_2 .$$

This is the general solution to the equation $u'' + r^{-1}u' = 1$. Now, since we want a solution which is continuous even at the origin $r = 0$ we choose $C_1 = 0$. Thus, our regular general solution is:

$$u(r) \;=\; \frac{1}{4}r^2 + C \; . \tag{2}$$

Here $C$ is any constant. We choose $C$ such that $u(a) = 0$, which means $C = -\frac{a^2}{4}$. Thus, our solution to the original problem is:

$$u(r) \;=\; \frac{1}{4}r^2 - \frac{a^2}{4} \; .$$

There is another somewhat shorter way to arrive at this solution, and in particular to get the general solution (2). What one could do is to first guess that solutions to $u'' + r^{-1}u' = 1$ should be of the form $C_1 r^\alpha$, for constants $C_1$ and $\alpha$. Upon substitution, the ODE becomes the algebraic equation:

$$C_1 \alpha^2 r^{\alpha-2} \;=\; 1 \; .$$

Thus, we just choose $\alpha = 2$ and $C_1 = \frac{1}{4}$. This is not good enough to get $u(a) = 0$, but we can easily add any constant to this and still get a solution to the ODE. Thus, we again arrive at the general form (2), from which we can derive the desired solution.

## Section 6.1, problem 9.

a) After making a reduction to the radial variable (again the boundary conditions are symmetric) this question asks to find the solution $u$ to the ODE problem:

$$\begin{aligned} \partial_r^2 u + \frac{2}{r}\partial_r u \;&=\; 0 \; , \\ u(1) \;&=\; 100 \; , \\ \partial_r u(2) \;&=\; -\gamma \; . \end{aligned}$$

The general solution to this radial ODE was derived in class and is $u(r) = \frac{C_1}{r} + C_2$. The second boundary condition gives us that $C_1 = 4\gamma$. Thus $u(r) = \frac{4\gamma}{r} + C$ for some $C$.

The first (inner) boundary condition then gives us that $C = 100 - 4\gamma$. Therefore, the desired solution is:

$$u(r) = \frac{4\gamma(1-r)}{r} + 100 \; .$$

b) In this setup, the temperature goes up as $r$ decreases and vice-versa. Thus, the hottest temperature is when $r = 1$ and is $100^o C$. The coldest temperature is when $r = 2$ and is $(100 - 2\gamma)^o C$.

Notice that this makes sense because there is a net flux *out* of the shell at the outer boundary ($\partial_r u = -\gamma < 0$ there). Thus there must be a net heat flux *into* the inner boundary, and therefore the temperature it hotter there.

c) Yes, all you have to do is to choose $\gamma = 40$ in the equation $(100 - 2\gamma)$ for the temperature on the outer boundary.

## Section 6.2, problem 3.

For this problem, we look for a solution which is obtained through separation of variables. That is, we look for a solution of the form:

$$u(x,y) = \sum_n X_n(x) Y_n(y) \ . \tag{3}$$

Since the only non-zero boundary condition here is in the $y$ variable, we will choose:

$$Y_0 = \frac{1}{2} \ , \qquad\qquad Y_2 = \frac{1}{2}\cos(2y) \ .$$

Therefore, we need to simply choose our $X_n$ such that they solve the following:

$$\begin{aligned} X_n'' &= n^2 X_n \ , \\ X_n(0) &= 0 \ , \\ X_n(\pi) &= 1 \ . \end{aligned}$$

If we do this, then we clearly get the correct solution by sticking things together according to formula (3). Solving the $X_n$ system for $n = 0$ we have $X_0(x) = \frac{x}{\pi}$. This is easily seen because $X_0'' = 0$ means that $X_0$ is a linear function $X_0 = a + bx$, and $a, b$ are determined by the endpoint conditions to be $a = 0$ and $b = \frac{1}{\pi}$.

Likewise, we solve for $X_2(x)$ by in general writing:

$$X_2(x) = a\cosh(2x) + b\sinh(2x) \ .$$

Choosing $a, b$ so that $X_2(0) = 0$ and $X_2(\pi) = 1$ we have that $a = 0$ and $b = \frac{1}{\sinh(2\pi)}$. Thus, we have that $X_2(x) = \frac{\sinh(2x)}{\sinh(2\pi)}$.

Adding all of the pieces together, we have that:

$$u(x,y) = \frac{x}{2\pi} + \frac{\sinh(2x)}{2\sinh(2\pi)}\cos(2y) \ .$$

## Section 6.2, problem 4 replacement.

To handle this problem, we first decompose it into two problems. The first is one where the boundary conditions are linear and give the correct values (same as the original problem) at the corners $(0,0)$, $(0,1)$, $(2,1)$, and $(2,0)$. That is, we first solve the problem:

$$\begin{aligned} \Delta u_0 &= 0 \ , & & \\ u_0(x,0) &= x \ , & \qquad u_0(x,1) &= 0 \ , \\ u_0(0,y) &= 0 \ , & \qquad u_0(2,y) &= 2(1-y) \ . \end{aligned}$$

This solution is easily computed to be $u_0 = x(1-y)$.

Next, we compute the solution to the continuous but non-linear boundary values problem:

$$\begin{aligned} \Delta u_1 &= 0 \ , & & \\ u_1(x,0) &= \sin\left(\frac{\pi}{2}x\right) \ , & u_1(x,1) &= 0 \ , \\ u_1(0,y) &= 0 \ , & u_1(2,y) &= 0 \ . \end{aligned}$$

In this case, we again proceed via separation of variables. Here we choose $X_1(x) = \sin\left(\frac{\pi}{2}x\right)$, so the corresponding eigenfunction $Y_1(y)$ must solve the system:

$$\begin{aligned} Y_1'' &= (\frac{\pi}{2})^2 Y_1 \ , \\ Y_1(0) &= 1 \ , \\ Y_1(1) &= 0 \ . \end{aligned}$$

That is, we have $Y_1(y) = a\cosh\left(\frac{\pi}{2}y\right) + b\sinh\left(\frac{\pi}{2}y\right)$. The boundary conditions easily give us:

$$\begin{aligned} a &= 1 \ , \\ b &= -\frac{\cosh\left(\frac{\pi}{2}\right)}{\sinh\left(\frac{\pi}{2}\right)} \ = \ -\coth\left(\frac{\pi}{2}\right) \ . \end{aligned}$$

Adding all of these things together, we see that the solution to the full problem may be written as:

$$u(x,y) \ = \ x(1-y) + \Big(\cosh\left(\frac{\pi}{2}y\right) - \coth\left(\frac{\pi}{2}\right)\sinh\left(\frac{\pi}{2}y\right)\Big)\sin\left(\frac{\pi}{2}x\right) \ .$$

## Section 6.3, problem 2.

This problem is a more or less direct use of Formula (10) on p. 160 of the text. We more or less immediately have that:

$$u(r,\theta) \ = \ 1 + \frac{3r}{a}\sin(\theta) \ .$$

## Section 6.3, problem 3.

In this problem, the series expansion of the boundary condition is given to us as $u(a,\theta) = 3\sin(\theta) - 3\sin(3\theta)$. Therefore, from our general formula for harmonic functions inside the disk, we have that:

$$u(r,\theta) = \frac{3r}{a}\sin(\theta) - \frac{3r^3}{a^3}\sin(3\theta) \ .$$

Its that simple.