MATH 110
PROOF OF UNIFORM CONVERGENCE OF FOURIER SERIES

Abstract. These notes are a supplement for Monday's lecture (11/20/06) on the uniform convergence of Fourier series.

The main result we are trying to prove is the following:

Theorem 0.1. *Suppose that $f(x)$ is a continuous periodic function with period 2π such that $\partial_x f(x)$ is also continuous and periodic. Then the Fourier series:*

$$f(x) \ = \ C + \sum_{n=1}^{\infty} \left(A_n \cos(nx) + B_n \sin(nx) \right) \ ,$$

where we define:

$$\begin{aligned} C \ &= \ \frac{1}{2\pi} \int_{-\pi}^{\pi} \ f(y) \ dy \ , \\ A_n \ &= \ \frac{1}{\pi} \int_{-\pi}^{\pi} \ \cos(ny) f(y) \ dy \ , \\ B_n \ &= \ \frac{1}{\pi} \int_{-\pi}^{\pi} \ \sin(ny) f(y) \ dy \ , \end{aligned}$$

converges uniformly *to $f(x)$ on $[-\pi, \pi]$.*

Proof. The statement of the theorem is that:

$$\lim_{N\to\infty} \sup_{-\pi\leqslant x\leqslant\pi} \Big| f(x) - \big[C + \sum_{n=1}^{N} \left(A_n \cos(nx) + B_n \sin(nx) \right) \big] \Big| \ \to \ 0 \ . \tag{1}$$

Recall from class that the proof essentially boils down to explicitly computing the partial Fourier series:

$$F_N(x) \ = \ C + \sum_{n=1}^{N} \left(A_n \cos(nx) + B_n \sin(nx) \right) \ .$$

After an explicit calculation which was done in class (this is also on p. 133 of the text), we have that:

$$F_N(x) \ = \ \int_{-\pi}^{\pi} \ Q_N(x-y) f(y) \ dy \ ,$$

where the kernel function $Q_N(z)$ is given by the explicit formula:

$$Q_N(z) \ = \ \frac{1}{2\pi} \cdot \frac{\sin\left((N+\frac{1}{2}) z \right)}{\sin\left(\frac{1}{2} z \right)} \ .$$

Recall also that another formula for $Q_N(z)$ was:

$$Q_N(z) \ = \ \frac{1}{2\pi}\sum_{n=-N}^{N} \ \cos(nz) \ ,$$

so that we always have:

$$\int_{-\pi}^{\pi} Q_N(x-y) \ dy \ = \ 1 \ ,$$

where $x \in [-\pi, \pi]$ is fixed. Thus, a way to write the sum on line (1) is as follows:

$$\Big| f(x) - \big[C + \sum_{n=1}^{N} (A_n \cos(nx) + B_n \sin(nx))\big]\Big| \ = \ \Big|\int_{-\pi}^{\pi} Q_N(x-y)\big(f(x)-f(y)\big) \ dy\Big| \ .$$

Therefore, we are trying to show that (a restatement of line (1)):

$$\lim_{N\to\infty} \ \sup_{-\pi\leqslant x\leqslant\pi} \Big|\int_{-\pi}^{\pi} Q_N(x-y)\big(f(x)-f(y)\big) \ dy\Big| \ \to \ 0 \ . \tag{2}$$

In order to simplify the proof, we will make one further reduction. This is based on the following identity, which follows from a simple change of variables $y' = y - x$ (note that Q_N is even):

$$\begin{aligned}\int_{-\pi}^{\pi} Q_N(x-y)\big(f(x)-f(y)\big) \ dy \ &= \ -\int_{-\pi-x}^{\pi-x} Q_N(y')\big(f(x+y')-f(x)\big) \ dy' \ , \\ &= \ -\int_{-\pi}^{\pi} Q_N(y)\big(f(x+y)-f(x)\big) \ dy \ .\end{aligned}$$

The last line above follows from the periodicity of the function $Q_N(y)\big(f(x+y) - f(x)\big)$ for each fixed $x \in [-\pi, \pi]$ (as a function of y). Therefore, substituting $L = N + \frac{1}{2}$ we see that it suffices to show:

$$\lim_{L\to\infty} \ \sup_{-\pi\leqslant x\leqslant\pi} \Big|\int_{-\pi}^{\pi} \sin\big(Ly\big)\cdot\frac{f(x+y)-f(x)}{\sin(\frac{1}{2}y)} \ dy\Big| \ \to \ 0 \ . \tag{3}$$

The proof is now centered around estimating the explicit integral on line (3). The goal is to show that if L is very large the entire expression is on the order $O(L^{-\frac{1}{3}})$. This will guarantee that as $L \to \infty$, the entire expression will become arbitrarily small.

To achieve this bound, we will chop the integral up into two pieces. The first is the "close region" where $|y| < L^{-\frac{1}{3}}$. This is where we cannot integrate by parts without causing trouble. Here we will simply use the fact that the interval we are integrating over is small to achieve the desired result. The other region is the "far region" where $\{|y| \geqslant L^{-\frac{1}{3}}\} \cap [-\pi, \pi]$. Here we will need to integrate by parts to pick up a small factor. Finally, keep in mind that $x \in [-\pi, \pi]$ is fixed, although we need to be careful and get bounds which are *uniform* in $x \in [-\pi, \pi]$ (that is, everything should not really depend on any fixed choice of x).

In the region $|y| < L^{-\frac{1}{3}}$ we are trying to estimate the integral:

$$\begin{aligned} &\Big|\int_{-L^{-\frac{1}{3}}}^{L^{-\frac{1}{3}}} \sin\big(Ly\big)\cdot\frac{f(x+y)-f(x)}{\sin(\frac{1}{2}y)}\ dy\Big| \ , \\ \leqslant\ &\int_{-L^{-\frac{1}{3}}}^{L^{-\frac{1}{3}}} \Big|\frac{f(x+y)-f(x)}{\sin(\frac{1}{2}y)}\Big|\ dy \ , \\ =\ &I(x,L) \ . \end{aligned}$$

This last line follows by first applying the so called "triangle inequality" for integrals $|\int_a^b f| \leqslant \int_a^b |f|$, and then simply using the fact that $|\sin\big(Ly\big)| \leqslant 1$. The main thing we need to do now is to show that the absolute integral $I(x,L)$ is small independent of the choice of x. In fact, we will show the bound:

$$I(x,L)\ \leqslant\ CL^{-\frac{1}{3}} \sup_{-\pi\leqslant x\leqslant\pi} |f'(x)| \ , \tag{4}$$

where C is a universal constant not depending on f or x. This suffices to prove that the "close part" of the integral on line (3) becomes smaller and smaller, because since $f'(x)$ is continuous it *must* be bounded on the interval $[-\pi,\pi]$. Of course this bound could be extremely large, but it has nothing to do with L so we will still get that things are going to zero as $L\to\infty$.

To show (4), it clearly suffices to be able to establish that:

$$\sup_{-\pi\leqslant y\leqslant\pi}\Big|\frac{f(x+y)-f(x)}{\sin(\frac{1}{2}y)}\Big|\ \leqslant\ \frac{C}{2}\sup_{-\pi\leqslant x\leqslant\pi}|f'(x)| \ . \tag{5}$$

Notice that (4) follows immediately from (5) because the domain of integration of $I(x,L)$ is an interval of length $2L^{-\frac{1}{3}}$. We have now reduced this portion of the proof to showing that (5) holds. But this bound just turns out to be the mean value theorem. To see this, we can first rewrite:

$$\Big|\frac{f(x+y)-f(x)}{\sin(\frac{1}{2}y)}\Big|\ =\ \Big|\frac{f(x+y)-f(x)}{y}\Big|\cdot\Big|\frac{y}{\sin(\frac{1}{2}y)}\Big| \ .$$

By L'Hôpital's rule, we have that when $|y|\leqslant\pi$ the following bound holds:

$$\Big|\frac{y}{\sin(\frac{1}{2}y)}\Big|\ \leqslant\ \frac{C}{2} \ ,$$

for some constant C (you could probably take $C = 10$, and can certainly take $C = 100$). Also, by the mean value theorem we have that:

$$\Big|\frac{f(x+y)-f(x)}{y}\Big|\ =\ |f'(\xi)| \ ,$$

for some $\xi\in[x,y]$. Therefore we easily have that:

$$\sup_{-\pi\leqslant y\leqslant\pi}\Big|\frac{f(x+y)-f(x)}{y}\Big|\ \leqslant\ \sup_{-\pi\leqslant x\leqslant\pi}|f'(x)| \ .$$

Multiplying this by the previous bound, we have (5).

To finish things off here, we need to be able to bound the integral on line (3) over the "far" region where $|y|\geqslant L^{-\frac{1}{3}}$ and $y\in[-\pi,\pi]$. Here we will rely on the oscillatory nature of the kernel Q_N for large N to provide decay. That is, we will simply integrate by parts to show that things are small. The integral on line (3) breaks up in this region into the sum of I_1 and I_2:

$$I_1 \ = \ \int_{L^{-\frac{1}{3}}}^{\pi} \sin\big(Ly\big)\cdot \frac{f(x+y)-f(x)}{\sin(\frac{1}{2}y)}\ dy \ ,$$

$$I_2 \ = \ \int_{-\pi}^{-L^{-\frac{1}{3}}} \sin\big(Ly\big)\cdot \frac{f(x+y)-f(x)}{\sin(\frac{1}{2}y)}\ dy \ .$$

We will only deal with the integral I_1 here, as the argument for I_2 is essentially the same. Integrating by parts we have that:

$$\begin{aligned} I_1 \ &= \ -\frac{1}{L}\cos(Ly)\cdot \frac{f(x+y)-f(x)}{\sin(\frac{1}{2}y)}\Big|_{L^{-\frac{1}{3}}}^{\pi} \\ &\qquad + \frac{1}{L}\int_{L^{-\frac{1}{3}}}^{\pi} \cos\big(Ly\big)\cdot \partial_y\left(\frac{f(x+y)-f(x)}{\sin(\frac{1}{2}y)}\right)\ dy \ , \\ &= \ I_{11} + I_{12} \ . \end{aligned} \tag{6}$$

The term I_{11} is easily bounded as follows:

$$|I_{11}| \ \leqslant \ CL^{-1}\sup_{-\pi\leqslant x\leqslant\pi}|f'(x)| \ , \tag{7}$$

by just using a reasoning almost identical to what has already been done in the "close" region above. Notice that the main thing is to make use of the uniform (in x) bound (5).

To bound the term I_{12} on line (6) we use the estimate:

$$\Big|\partial_y\left(\frac{f(x+y)-f(x)}{\sin(\frac{1}{2}y)}\right)\Big| \ \leqslant \ \Big|\frac{f'(x+y)}{\sin(\frac{1}{2}y)}\Big| + \frac{1}{2}\Big|\frac{f(x+y)-f(x)}{\sin^2(\frac{1}{2}y)}\Big| \ .$$

Using now the fact that $|y| \geqslant L^{-\frac{1}{3}}$ we (after a little easy work) have the bound:

$$\Big|\partial_y\left(\frac{f(x+y)-f(x)}{\sin(\frac{1}{2}y)}\right)\Big| \ \leqslant \ CL^{\frac{2}{3}}\left(\sup_{-\pi\leqslant x\leqslant\pi}|f(x)| + \sup_{-\pi\leqslant x\leqslant\pi}|f'(x)|\right) \ ,$$

where C is some *fixed* explicit constant which can be computed without too much trouble (e.g. $C = 100$ is good enough). Therefore, we see that we can bound the integral I_{12} as follows:

$$\begin{aligned} |I_{12}| \ &\leqslant \ \frac{1}{L}\int_{L^{-\frac{1}{3}}}^{\pi}\Big|\partial_y\left(\frac{f(x+y)-f(x)}{\sin(\frac{1}{2}y)}\right)\Big|\ dy \ , \\ &\leqslant \ \frac{1}{L}\cdot CL^{\frac{2}{3}}\left(\sup_{-\pi\leqslant x\leqslant\pi}|f(x)| + \sup_{-\pi\leqslant x\leqslant\pi}|f'(x)|\right)\int_{L^{-\frac{1}{3}}}^{\pi} dy \ , \\ &\leqslant \ L^{-\frac{1}{3}}C\pi\left(\sup_{-\pi\leqslant x\leqslant\pi}|f(x)| + \sup_{-\pi\leqslant x\leqslant\pi}|f'(x)|\right) \ . \end{aligned}$$

Adding together the bound on this last line with the estimate on line (7) above, we have:

$$|I_1| \ \leqslant \ L^{-\frac{1}{3}}C2\pi\left(\sup_{-\pi\leqslant x\leqslant\pi}|f(x)| + \sup_{-\pi\leqslant x\leqslant\pi}|f'(x)|\right) \ .$$

By an identical argument we will also ave that:

$$|I_2| \ \leqslant \ L^{-\frac{1}{3}} C 2\pi \left(\sup_{-\pi\leqslant x\leqslant\pi} |f(x)| + \sup_{-\pi\leqslant x\leqslant\pi} |f'(x)| \right) \ .$$

Finally, we may add the previous two bounds to the estimate from line (5) for the "close part" of the total integral. Doing this, we end up with the explicit bound:

$$\sup_{-\pi\leqslant x\leqslant\pi} \Big| \int_{-\pi}^{\pi} \sin\left(Ly\right) \cdot \frac{f(x+y)-f(x)}{\sin(\frac{1}{2}y)} \, dy \Big| \ \leqslant \ L^{-\frac{1}{3}} C 5\pi \left(\sup_{-\pi\leqslant x\leqslant\pi} |f(x)| + \sup_{-\pi\leqslant x\leqslant\pi} |f'(x)| \right) \ ,$$

where C is again an explicit constant which could have been computed with a little more trouble (just take $C = 100$ to be safe). The point is that all of the constants *don't* depend on L. Therefore, we easily have that as $L \to \infty$, the sup in x of the absolute value goes to zero. This is precisely the statement of uniform convergence.

One thing to notice in this proof is that our estimates will get worse and worse for functions $f(x)$ such that $f'(x)$ is very large. This is not an "artifact" of the proof, but rather points to a central difficulty in the summation of Fourier series. As we have mentioned before, if one drops the condition that $f'(x)$ is bounded and just looks at continuous $f(x)$, then uniform convergence of the corresponding Fourier series can actually break down. One can construct examples of continuous functions such that their Fourier series *does not* converge uniformly, and does not even converge pointwise at "many" points.

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