MATH 110
EXAM #2

Please answer the following questions. Because this test is open book and open note, you *will not get credit* for answers unless you demonstrate how you arrived at them. In short, please show all work.

PROBLEM 1.

Consider the following function $\phi(x)$ defined on the interval $(-1, 1)$:

$$\phi(x) \;=\; \begin{cases} 1\ , & 0 \;<\; x \;<\; 1\ ; \\ 0\ , & x \;=\; 0\ ; \\ -1\ , & -1 \;<\; x \;<\; 0\ . \end{cases}$$

Here is a sketch of the graph:

Please compute the full sine/cosine Fourier series of $\phi(x)$ on the interval $[-1, 1]$.

Problem 2.

Consider the following periodic "sawtooth" function $\psi(x)$:

$$\psi(x) \;=\; \begin{cases} x - 2n \;, & 2n \;\leqslant\; x \;<\; 2n+1 \;; \\ -x + 2n + 2 \;, & 2n+1 \;\leqslant\; x \;<\; 2n+2 \;. \end{cases}$$

for all $n \in \mathbb{Z}$. Here is a picture:

Please answer the following questions:

a) Because of the sharp turns of this function, the derivative $\partial_x \psi(x)$ will not be defined on the integers $x = n \in \mathbb{Z}$. However, in each of the open intervals $(n, n+1)$ the function $\partial_x \psi(x)$ is well defined. Compute the function $\partial_x \psi(x)$, except for at the points $x = n \in \mathbb{Z}$.

b) Compute the full sine/cosine series of the periodic function $\psi(x)$. (Hint: It may be useful to compare the answer from part a) with the computation you did in problem 1) above. You may assume that Fourier series can be integrated term by term.)

Problem 3.

Consider the following heat flow problem in a uniform metal bar of length π. The bar is heated at one end (say $x = 0$) at the constant temperature $T = 1$. The other end of the bar is insulated (say at $x = \pi$). Supposing that the diffusion constant for this problem is $k = 1$, the PDE problem for the temperature function $u(t, x)$ is the following:

$$\begin{aligned} u_t &= u_{xx} , & x \in [0, \pi] , \\ u(t, 0) &= 1 , \\ u_x(t, \pi) &= 0 , \\ u(0, x) &= f(x) . \end{aligned}$$

Please answer the following questions:

a) Suppose one were to try to set up an eigenvalue problem for the separation of variables here. Then we would be trying to solve:

$$\begin{aligned} -X''(x) &= \lambda X(x) , & x \in [0, \pi] , \\ X(0) &= 1 , \\ X'(\pi) &= 0 . \end{aligned}$$

Are these boundary conditions symmetric? (Hint: If you forgot, this notion is discussed on p. 115 of the text, towards the bottom of the page.)

b) Find the solution $u(t, x)$ in the case where the initial data $f(x)$ is the function:

$$f(x) \ = \ 1 + \sin\big(\frac{1}{2}x\big) - \sin\big(\frac{5}{2}x\big) \ .$$

(Hints: It will be useful for you to notice the following things:

- Consider the auxiliary function $w(t, x) = u(t, x) - 1$. What type of equation, boundary conditions, and initial data does this new function satisfy?

- Recall from Section 4.2 problem 1 that the eigenvalue problem:

  $$\begin{aligned} -X''(x) &= \lambda X(x) , & x \in [0, \pi] , \\ X(0) &= X'(\pi) \ = \ 0 , \end{aligned}$$

  has solutions $X_n(x) = \sin\big((n + 1/2)x\big)$, with eigenvalues $\lambda_n = (n + 1/2)^2$, for integers $n \in \mathbb{Z}$.)

c) As time $t \to \infty$, what is the steady-state temperature T_0 that is approached. That is, what is the limit $T_0 = \lim_{t\to\infty} u(t, x)$? Why does this answer make sense?