MATH 110
SOLUTIONS TO EXAM #2

Problem 1.

Consider the following function $\phi(x)$ defined on the interval $(-1,1)$:

$$\phi(x) \;=\; \begin{cases} 1\ , & 0 \;<\; x \;<\; 1\ ; \\ 0\ , & x \;=\; 0\ ; \\ -1\ , & -1 \;<\; x \;<\; 0\ . \end{cases}$$

Please compute the full sine/cosine Fourier series of $\phi(x)$ on the interval $[-1,1]$.

Solution. The first thing to notice here is that this function is *odd*. Therefore, all of the cosine terms in it Fourier expansion must be zero. Specifically, if we write $\phi(x) = \frac{1}{2}A_0 + \sum_{0<n}\big(A_n\cos(n\pi x) + B_n\sin(n\pi x)\big)$ we have that:

$$A_n \;=\; \int_{-1}^{1}\phi(x)\cos(n\pi x)\ dx \;=\; 0\ .$$

Therefore, it remains to compute the B_n coefficients. These are given as follows:

$$\begin{aligned} B_n \;&=\; \int_{-1}^{1}\phi(x)\sin(n\pi x)\ dx\ , \\ &=\; 2\int_{0}^{1}\phi(x)\sin(n\pi x)\ dx\ , \\ &=\; 2\int_{0}^{1}\sin(n\pi x)\ dx\ , \\ &=\; -\frac{2}{n\pi}\big(\cos(n\pi x)\big)\Big|_0^1\ , \\ &=\; -\frac{2}{n\pi}\big((-1)^n - 1\big)\ . \end{aligned}$$

Therefore, we see that:

$$B_n \;=\; \begin{cases} \frac{4}{n\pi}\ , & n \text{ is odd;} \\ 0\ , & n \text{ is even.} \end{cases}$$

Putting all of this together, we see that the series expansion for $\phi(x)$ is the following:

$$\phi(x) \;=\; \frac{4}{\pi}\sum_{0<n\quad odd}\frac{1}{n}\sin(n\pi x)\ .$$

Problem 2.

Consider the following periodic "sawtooth" function $\psi(x)$:

$$\psi(x) \ = \ \begin{cases} x - 2n \ , & 2n \ \leqslant \ x \ < \ 2n+1 \ ; \\ -x + 2n + 2 \ , & 2n+1 \ \leqslant \ x \ < \ 2n+2 \ . \end{cases}$$

for all $n \in \mathbb{Z}$. Please answer the following questions:

a) Because of the sharp turns of this function, the derivative $\partial_x \psi(x)$ will not be defined on the integers $x = n \in \mathbb{Z}$. However, in each of the open intervals $(n, n+1)$ the function $\partial_x \psi(x)$ is well defined. Compute the function $\partial_x \psi(x)$, except for at the points $x = n \in \mathbb{Z}$.

Solution. This is a simple computation. We have that:

$$\partial_x \psi(x) \ = \ \begin{cases} 1 \ , & 2n \ < \ x \ < \ 2n+1 \ ; \\ -1 \ , & 2n+1 \ < \ x \ < \ 2n+2 \ . \end{cases}$$

Thus, one can consider $\phi(x) = \partial_x \psi(x)$ as the periodic extension of the function ϕ from the previous problem.

b) Compute the full sine/cosine series of the periodic function $\psi(x)$. (Hint: It may be useful to compare the answer from part a) with the computation you did in problem 1) above. You may assume that Fourier series can be integrated term by term.)

Solution. From the previous problem we have the series expansion:

$$\partial_x \psi(x) \ = \ \phi(x) \ = \ \frac{4}{\pi} \sum_{0<n \quad odd} \frac{1}{n} \sin(n\pi x) \ .$$

Taking the antiderivative of this sum term by term, we have:

$$\psi(x) \ = \ C - \frac{4}{\pi^2} \sum_{0<n \quad odd} \frac{1}{n^2} \cos(n\pi x) \ ,$$

For some constant C. Recall that the constant term in a Fourier series is given by the average of the function over its period. That is:

$$\begin{aligned} C \ &= \ \frac{1}{2} \int_{-1}^{1} \ \psi(x) \ dx \ , \\ &= \ \int_0^1 x \ dx \ , \\ &= \ \frac{1}{2} \ . \end{aligned}$$

Therefore, we have that:

$$\psi(x) \ = \ \frac{1}{2} - \frac{4}{\pi^2} \sum_{0<n \quad odd} \frac{1}{n^2} \cos(n\pi x) \ .$$

Problem 3.

Consider the following heat flow problem in a uniform metal bar of length π. The bar is heated at one end (say $x = 0$) at the constant temperature $T = 1$. The other end of the bar is insulated (say at $x = \pi$). Supposing that the diffusion constant for this problem is $k = 1$, the PDE problem for the temperature function $u(t, x)$ is the following:

$$\begin{aligned} u_t &= u_{xx} \ , \qquad\qquad x \in [0, \pi] \ , \\ u(t, 0) &= 1 \ , \\ u_x(t, \pi) &= 0 \ , \\ u(0, x) &= f(x) \ . \end{aligned}$$

Please answer the following questions:

a) Suppose one were to try to set up an eigenvalue problem for the separation of variables here. Then we would be trying to solve:

$$\begin{aligned} -X''(x) &= \lambda X(x) \ , \qquad\qquad x \in [0, \pi] \ , \\ X(0) &= 1 \ , \\ X'(\pi) &= 0 \ . \end{aligned}$$

Are these boundary conditions symmetric? (Hint: If you forgot, this notion is discussed on p. 115 of the text, towards the bottom of the page.)

Solution. It is *not* symmetric. Recall that this means $(f'g - fg')|_0^\pi = 0$ for any par of function f, g satisfying the boundary conditions. In this case it would mean that:

$$f'(0) - g'(0) \ = \ 0 \ ,$$

for any par of functions obeying the boundary conditions. Its not hard to see that this cannot hold in general.

b) Find the solution $u(t, x)$ in the case where the initial data $f(x)$ is the function:

$$f(x) \ = \ 1 + \sin\big(\frac{1}{2}x\big) - \sin\big(\frac{5}{2}x\big) \ .$$

(Hints: It will be useful for you to notice the following things:

- Consider the auxiliary function $w(t, x) = u(t, x) - 1$. What type of equation, boundary conditions, and initial data does this new function satisfy?

- Recall from Section 4.2 problem 1 that the eigenvalue problem:

$$\begin{aligned} -X''(x) &= \lambda X(x) \ , \qquad\qquad x \in [0, \pi] \ , \\ X(0) &= X'(\pi) \ = \ 0 \ , \end{aligned}$$

  has solutions $X_n(x) = \sin\big((n + 1/2)x\big)$, with eigenvalues $\lambda_n = (n + 1/2)^2$, for integers $n \in \mathbb{Z}$.)

Solution. The way to do this calculation is the following. First we form the function $w = u - 1$. This new function satisfies the mixed Dirichlet-Neumann boundary value problem:

$$\begin{aligned} w_t &= w_{xx} \ , & x \in [0, \pi] \ , \\ w(t,0) &= 0 \ , \\ w_x(t,\pi) &= 0 \ , \\ w(0,x) &= \sin\big(\frac{1}{2}x\big) - \sin\big(\frac{5}{2}x\big) \ . \end{aligned}$$

The solution of this is easily computed to be:

$$w(t,x) \ = \ e^{-\frac{1}{4}t}\sin\big(\frac{1}{2}x\big) - e^{-\frac{25}{4}t}\sin\big(\frac{5}{2}x\big) \ .$$

Therefore, we have that the original solution is:

$$u(t,x) \ = \ 1 + e^{-\frac{1}{4}t}\sin\big(\frac{1}{2}x\big) - e^{-\frac{25}{4}t}\sin\big(\frac{5}{2}x\big) \ .$$

c) As time $t \to \infty$, what is the steady-state temperature T_0 that is approached. That is, what is the limit $T_0 = \lim_{t\to\infty} u(t,x)$? Why does this answer make sense?

Solution. We see that as time $t \to \infty$, we have that:

$$\begin{aligned} T_0 &= \lim_{t\to\infty} \Big(1 + e^{-\frac{1}{4}t}\sin\big(\frac{1}{2}x\big) - e^{-\frac{25}{4}t}\sin\big(\frac{5}{2}x\big)\Big) \ , \\ &= 1 + \lim_{t\to\infty} e^{-\frac{1}{4}t}\sin\big(\frac{1}{2}x\big) + \lim_{t\to\infty} e^{-\frac{25}{4}t}\sin\big(\frac{5}{2}x\big) \ , \\ &= 1 + 0 + 0 \ = \ 1 \ . \end{aligned}$$

This makes sense because the endpoint of the rod is being heated at a constant temperature equal to 1, and the insulation at the other end makes sure no excess heat is escaping. Thus, eventually the rod should have a uniform temperature T_0 equal to the temperature at the heated end.