MATH 110
SOLUTIONS TO HW #7

Section 6.1, Problem 6

The way we will solve this is the following. Notice that since the boundary conditions are rotationally symmetric, the solution must be also. Therefore, we assume that the solution is a function $u(r)$ which solves the equation:

$$\partial_r^2 u + \frac{1}{r}\partial_r u \ = \ 1 \ ,$$
$$u(a) \ = \ u(b) \ = \ 0 \ .$$

We solve the ODE first. This may be written in the form:

$$\partial_r(r u_r) \ = \ r \ .$$

Therefore, we have:

$$u_r \ = \ \frac{1}{2}r + \frac{A}{r} \ .$$

This gives the general solution:

$$u(r) \ = \ \frac{1}{4}r^2 + A\ln(r) + B \ .$$

to solve for the boundary conditions, we end up with the system of equations:

$$0 \ = \ \frac{1}{4}a^2 + A\ln(a) + B \ ,$$
$$0 \ = \ \frac{1}{4}b^2 + A\ln(b) + B \ .$$

This is a linear system of two equations in two unknowns. The solution is:

$$A \ = \ \frac{1}{4\big(\ln(a) - \ln(b)\big)}(b^2 - a^2) \ ,$$
$$B \ = \ \frac{1}{4\big(\ln(a) - \ln(b)\big)}\big(a^2\ln(b) - b^2\ln(a)\big) \ .$$

Section 6.1, Problem 10

For this problem, we set $w = u - v$, then:

(1) $$\Delta w \ = \ 0 \ ,$$

(2) $$w|_{\partial\Omega} \ = \ 0 \ .$$

The goal is now to show that $w \equiv 0$. To do this, we employ the following Green's identity:

$$\iint_\Omega |\nabla w|^2 \ dxdy \ = \ -\iint_\Omega w\Delta w \ dxdy + \int_{\partial\Omega} \frac{\partial w}{\partial \hat{n}} w \ d\ell \ .$$

In the case where both (1)–(2) hold, we have from the above formula that:

$$\iint_\Omega |\nabla w|^2 \ dxdy \ = \ 0 \ .$$

This in turn implies that $\nabla w \equiv 0$. Thus $w \equiv C$ for some constant C, which gives $w \equiv 0$ from the boundary conditions.

Section 6.1, Problem 11

This is from a direct integration of the quantity $\nabla \cdot \nabla u = f$ and using the divergence theorem:

$$\begin{aligned} \iiint_\Omega f \ dxdydz \ &= \ \iiint_\Omega \nabla \cdot \nabla u \ dxdydz \ , \\ &= \ \iint_{\partial\Omega} \hat{n} \cdot \nabla u \ dS \ , \\ &= \ \iint_{\partial\Omega} g \ dS \ . \end{aligned}$$

Typed problem # 2

A solution to this problem is of the form:

$$u(r) \ = \ A_0 + \sum_{1 \leqslant n} r^n \big(A_n \cos(n\theta) + B_n \sin(n\theta) \big) \ .$$

The first boundary condition gives $u = 1$. Of course, this makes the second boundary condition impossible.

Typed problem # 3

The solution is:

$$u(r, \theta) \ = \ 1 - 3r^3 \cos(3\theta) + 2r^5 \sin(5\theta) \ .$$

Typed problem # 4

Again, the general solution to $\Delta u = 0$ is:

$$u(r,\theta) \;=\; \widetilde{A}_0 + \sum_{1\leqslant n} r^n \big(\widetilde{A}_n \cos(n\theta) + \widetilde{B}_n \sin(n\theta)\big) \ .$$

This leads to the formula:

$$f(\theta) \;=\; (\partial_r u)(R,\theta) \;=\; \sum_{1\leqslant n} nR^{n-1} \big(\widetilde{A}_n \cos(n\theta) + \widetilde{B}_n \sin(n\theta)\big) \ .$$

Suppose now that:

$$f(\theta) \;=\; A_0 + \sum_{1\leqslant n} r^n \big(A_n \cos(n\theta) + B_n \sin(n\theta)\big) \ .$$

Matching terms, this gives the formulas:

$$\begin{aligned} \widetilde{A}_0 &= 0 \ , \\ \widetilde{A}_n &= \frac{A_n}{nR^{n-1}} \ , \\ \widetilde{B}_n &= \frac{B_n}{nR^{n-1}} \ . \end{aligned}$$

In other words, we have the formula:

$$u(r,\theta) \;=\; \sum_{1\leqslant n} \frac{r^n}{nR^{n-1}} \big(A_n \cos(n\theta) + B_n \sin(n\theta)\big) \ .$$

Typed problem # 5

Using the last formula, we have:

$$u(r,\theta) \;=\; \frac{3}{2} r^2 \cos(2\theta) - \frac{1}{2^{10}} r^{10} \sin(10\theta) \ .$$