Homework for Math 110
Introduction to PDE
Fall 2006

Due Friday November 16:

(1) Please compute the Fourier sin series on the interval $[0,1]$ for the "saw-tooth" function:

$$f(x) \ = \ \begin{cases} \frac{3}{2}x \ , & \text{if } 0 \leqslant x \leqslant \frac{2}{3}; \\ 3-3x \ , & \text{if } \frac{2}{3} < x \leqslant 1. \end{cases}$$

(2) Please compute the Fourier cos series on the interval $[0,\pi]$ for the function:

$$f(x) \ = \ \cos^2(x) \ .$$

(Hint: It will be useful here to use some double angle identities).

(3) Section 5.1:

8 For this problem, you first need to subtract off the "equilibrium" solution u_0 which solves the problem: $\partial_x^2 u_0 \ = \ 0$ with $u_0(0) \ = \ 0$ and $u_0(1) \ = \ 1$. Then find the solution $w = u - u_0$ using the first problem above.

9 For this problem, simply use 2) above.

(4) Section 5.2: 2, 6, 12.

Due Monday December 3:

(1) Section 5.3: 2, 5 (use on # 3), 3, 10.
(2) Section 5.4: 1, 8a), 8b), 19.

Due Monday December 10:

(1) Section 6.1: 6, 10, 11.

(2) Show that there is *no* solution to the boundary value problem:

$$\begin{aligned} \Delta u &= 0 , && \text{in } r < 1 , \\ u|_{r=1} &= 1 , \\ \partial_r u|_{r=1} &= 1 . \end{aligned}$$

(Hint: Notice that radially symmetric boundary conditions imply that the solution u must also be radially symmetric. Simply use the explicit formula for radially symmetric solutions.) This exercise shows that having *both* Dirichlet and Neumann data is in general overdetermined, and there is no solution possible. This might seem a bit surprising at first because $\Delta u = 0$ is a second order equation, and one expect to have two boundary conditions like for the wave equation.

(3) Please compute the explicit solution to the following problem:

$$\begin{aligned} \Delta u &= 0 , && \text{in } r < 1 , \\ u|_{r=1} &= 1 - 3\cos(3\theta) + 2\sin(5\theta) . \end{aligned}$$

(4) In this problem you will write down a formula for the Neumann problem:

$$\begin{aligned} \Delta u &= 0 , && \text{in } r < R , \\ \partial_r u|_{r=R} &= f(\theta) . \end{aligned}$$

First, write down the general formula for a solution to $\Delta u = 0$ in $r < R$ (just the usual Poisson formula). Next, in terms of this general formula, compute:

$$f(\theta) = \partial_r u|_{r=R} .$$

Finally, suppose that $f(\theta)$ is given by:

$$f(\theta) = A_0 + \sum_{1 \leqslant n} \big(A_n \cos(n\theta) + B_n \sin(n\theta) \big) .$$

Compute the solution $u(r, \theta)$ in terms of the A_n, B_n.

(5) Use the last problem to compute the explicit solution to the problem:

$$\begin{aligned} \Delta u &= 0 , && \text{in } r < 2 , \\ \partial_r u|_{r=2} &= 6\cos(2\theta) - 5\sin(10\theta) . \end{aligned}$$