MATH 110
PRACTICE FINAL

Please answer the following questions. Because this test is take-home, you *will not get credit* for answers unless you demonstrate how you arrived at them. In short, please show all work.

PROBLEM 1.

Consider the following transport equation in the plane for the function $u(x, y)$:

$$\partial_x u + 2x\partial_y u \ = \ 0 \ . \tag{1}$$

a) Solve equation (1) with the following conditions on the y-axis:

$$u(0, y) \ = \ y^2 \ .$$

b) Is it possible to find a solution of (1) in the entire plane such that it has the following condition on the x-axis:

$$u(x, 0) \ = \ \sin(x) \ .$$

If not, give the reason why. (Hint: It may be helpful to sketch the characteristics of the equation (1)).

PROBLEM 2.

This problem is close to something we discussed in a lecture. Consider the wave equation for a vibrating string of length $\ell = \pi$ with fixed endpoints (i.e. the Dirichlet problem). Suppose that the wave speed is $c = 1$ (i.e. we have that $T = \rho = 1$ for the tension and mass density). Suppose that the initial Cauchy data is given by:

$$\begin{aligned} u(0,x) &= \sin(x) + 2\sin(3x) - 5\sin(6x) + \sin(11x) \ , \\ u_t(0,x) &= -\sin(x) + 4\sin(7x) + \sin(10x) - 3\sin(15x) \ . \end{aligned}$$

Compute the energy at each time t for the wave $u(t,x)$. That is, compute the integral:

$$E(t) \ = \ \frac{1}{2}\int_0^{\pi} \big(u_t^2(t,x) + u_x^2(t,x)\big) \ dx \ .$$

(Hint: There is an easy way and a hard way to do this. Orthogonality and the conservation of energy are your friends.)

PROBLEM 3.

Prove that the series from Problem #2 of the last exam converges uniformly. That is, prove that the Fourier series:

$$\psi(x) \ = \ \frac{1}{2} - \frac{4}{\pi^2} \sum_{0<n \ odd}^{\infty} \frac{1}{n^2} \cos(n\pi x) \ ,$$

converges uniformly on the interval $[-1,1]$. Specifically, show that by taking N_0 large enough, one can make the uniform error:

$$\sup_{-1\leqslant x\leqslant 1} \Big| \psi(x) - \Big(\frac{1}{2} - \frac{4}{\pi^2} \sum_{0<n \ odd}^{N} \frac{1}{n^2} \cos(n\pi x) \Big) \Big| \ = \ \mathcal{E}_N \ ,$$

as small as we like for all $N_0 \leqslant N$.

To show this, you may assume the following two facts:

- First, you may assume the following generalized triangle inequality:

  $$|\sum f_n| \ \leqslant \ \sum |f_n| \ ,$$

  where the sum has any number, including an infinite number, of terms (assuming the sum converges).
- Second, you may assume that one can make the infinite sum:

  $$S_{N_0} \ = \ \sum_{n\geqslant N_0} \frac{1}{n^2} \ ,$$

  as small as you like by taking N_0 large enough (its just the "tail" of a convergent infinite sum after all).

PROBLEM 4.

a) Solve the following Dirichlet problem in the unit circle:

$$\begin{aligned} \Delta u &= 0 , && \text{in } 0 \leqslant r < 1 , \\ u(1,\theta) &= 1 + \sin(2\theta) + \cos(2\theta) . \end{aligned}$$

b) Verify that the maximum principle is true for this explicit solution. (Hint: It might be easiest to first subtract off a constant. Notice that if u is any function, then $u - C$ will have its maximum at the *same* point that u does, and that the maximum of $u - C$ is just the maximum of u minus C.)