MATH 110
EXAM #1

Please answer the following questions. Because this test is open book and open note, you *will not get credit* for answers unless you demonstrate how you arrived at them. In short, please show all work.

PROBLEM 1.

Compute the specific solution $u(x, y)$ to the transport equation:

$$\frac{1}{1+x^2}\partial_x u + \frac{1}{1+y^2}\partial_y u \ = \ 0 \ ,$$

such that when $y = 0$ one has $u(x, 0) = x + \frac{1}{3}x^3$.

Solution: First we find an explicit formula for the general solution. The characteristic equation in this case is:

$$\begin{aligned}\frac{dy}{dx} \ &= \ \frac{\frac{1}{1+y^2}}{\frac{1}{1+x^2}} \ , \\ &= \ \frac{1+x^2}{1+y^2} \ .\end{aligned}$$

Separating the variables, this becomes:

$$(1+y^2)dy \ = \ (1+x^2)dx \ ,$$

which integrates to:

$$y + \frac{1}{3}y^3 \ = \ x + \frac{1}{3}x^3 + C \ ,$$

or solving for C:

$$y + \frac{1}{3}y^3 - x - \frac{1}{3}x^3 \ = \ C \ .$$

Therefore, the general solution to this problem looks like:

$$u(x, y) \ = \ F(y + \frac{1}{3}y^3 - x - \frac{1}{3}x^3) \ ,$$

for an arbitrary function F.

Setting $y = 0$ we get the specific solution be solving:

$$F(-x - \frac{1}{3}x^3) \ = \ x + \frac{1}{3}x^3 \ .$$

Therefore, we need to set $F(\xi) \ = \ -\xi$. Thus:

$$u(x, y) \ = \ -y - \frac{1}{3}y^3 + x + \frac{1}{3}x^3 \ .$$

Problem 2.

Consider the solution $u(t,x)$ to the wave equation:

$$u_{tt} - u_{xx} \ = \ 0 \ ,$$

with initial data:

$$u(0,x) \ = \ 0 \ ,$$
$$u_t(0,x) \ = \ \begin{cases} 1 \ , & |x| \ \leqslant \ 1 \ , \\ 0 \ , & \text{otherwise} \ . \end{cases}$$

Compute $\partial_x u$, and then fix $x = 0$. Show that as $t \to \infty$ we have:

$$\partial_x u(t,0) \ = \ 0 \ , \qquad\qquad t \text{ is large.}$$

Can you give an upper bound on t (for $0 \leqslant t$) such that the above identity holds? (Hint: Use the explicit formula for the solution. You may find it useful to recall the following general formula for definite integrals $\int_a^b = \int_a^0 + \int_0^b = \int_0^b - \int_0^a$.)

Solution: For this problem, lets agree to call the initial data for u_t by symbol:

$$u_t(x,0) \ = \ \chi_{[-1,1]}(x) \ .$$

With this designation, the explicit (D'Alembert) formula for the solution $u(x,t)$ looks like:

$$u(x,t) \ = \ \frac{1}{2} \int_{x-t}^{x+t} \ \chi_{[-1,1]}(s) \ ds \ .$$

Using the basic rules for integrals, this can be written as:

$$u(x,t) \ = \ \frac{1}{2}\Big(\int_0^{x+t} \ \chi_{[-1,1]}(s) \ ds - \int_0^{x-t} \ \chi_{[-1,1]}(s) \ ds \Big) \ .$$

Taking a time derivative of this last expression and using the fundamental theorem of calculus to compute that derivative, we have:

$$u_t(x,t) \ = \ \frac{1}{2}\Big(\chi_{[-1,1]}(x+t) + \chi_{[-1,1]}(x-t) \Big) \ .$$

Setting $x = 0$ in this last line, we finally arrive at the formula:

$$u_t(0,t) \ = \ \frac{1}{2}\Big(\chi_{[-1,1]}(t) + \chi_{[-1,1]}(-t) \Big) \ .$$

Making $1 < t$ in the above formula, we see that:

$$u_t(0,t) \ = \ 0 \ , \qquad\qquad 1 \ < \ t \ .$$

This follows easily from the definition of $\chi_{[-1,1]}(t)$, which shows that it is always zero for $1 < |t|$.

Problem 3.

Consider the heat flow for an insulated rod:

$$\partial_t u = \partial_x^2 u \ ,$$
$$\partial_x u(t,0) \ = \ \partial_x u(t,\ell) \ = \ 0 \ ,$$
$$u(0,x) \ = \ f(x) \ .$$

The "total heat" in the rod at time $t = 0$ is defined to be:

$$H(0) \ = \ \int_0^\ell \ f(x) \ dx \ .$$

That is, just the integral of the initial heat distribution over the rod. Show that in this case the total heat distribution remains *constant* for all times $0 \leqslant t$. That is, if we define:

$$H(t) \ = \ \int_0^\ell \ u(t,x) \ dx \ ,$$

then we must have $H(t) = H(0)$. (Hint: This is similar to the energy estimates we derived in class. The goal in this case is to show that $\frac{d}{dt}H(t) = 0$.)

Solution: This is a direct calculation. We compute the time derivative by "differentiation under the integral sign":

$$\begin{aligned} \frac{dH}{dt}(t) \ &= \ \int_0^\ell \ u_t(x,t) \ dx \ , \\ &= \ \int_0^\ell \ u_{xx}(x,t) \ dx \ , \\ &= \ u_x(x,t)|_0^\ell \ , \\ &= \ 0 - 0 \ , \\ &= \ 0 \ . \end{aligned}$$

Notice that the second to third line here follows from the fundamental theorem of calculus.