FACT SHEET FOR 20C EXAM 1

ABSTRACT. Here are some formulas that you should know for the first exam. Please bring a copy of this sheet to the test. Copies will **not** be included with the exams. Please do not write extra formulas on this sheet.

1. The Dot Product

- The "dot product" of two vectors is defined to be:

$$\vec{u}\cdot\vec{v} \;=\; u_1v_1 + u_2v_2 \ , \qquad \text{(In 2D)}$$
$$\vec{u}\cdot\vec{v} \;=\; u_1v_1 + u_2v_2 + u_3v_3 \ . \qquad \text{(In 3D)}$$

- The length of a vector may be written in terms of the dot product: $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$. More generally, the dot product gives combined information about both lengths and angles:

$$\vec{u}\cdot\vec{v} \;=\; \|\vec{u}\|\,\|\vec{v}\|\cos(\theta) \ ,$$

where θ is the angle between $\vec{u}$ and $\vec{v}$ in *radians*.

2. The Cross Product

- The "cross product" of two 3D vectors is given by the formula:

$$\vec{u}\times\vec{v} \;=\; (u_2v_3 - u_3v_2)\vec{i} + (u_3v_1 - u_1v_3)\vec{j} + (u_1v_2 - u_2v_1)\vec{k} \ .$$

The cross product is a vector, and it is *only* defined in 3D.
- The cross product always produces a vector perpendicular to both $\vec{u}$ and $\vec{v}$. That is:

$$(\vec{u}\times\vec{v})\cdot\vec{u} \;=\; 0 \ , \qquad (\vec{u}\times\vec{v})\cdot\vec{v} \;=\; 0 \ .$$

The length of the cross product obeys the sine rule:

$$\|\vec{u}\times\vec{v}\| \;=\; \|\vec{u}\|\,\|\vec{v}\|\sin(\theta) \ ,$$

where θ is the angle going from $\vec{u}$ to $\vec{v}$ according to the right hand rule, and by convention lies in the range $0 \leqslant \theta \leqslant \pi$.

3. Determinants

- In general, the three dimensional determinant is given by the formula:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \;=\; a(ei - hf) - b(di - gf) + c(dh - ge) \ .$$

- If $\vec{u}$, $\vec{v}$, $\vec{w}$ are any three vectors, then the *volume* of the parallelepiped spanned by them is equal to the absolute value $|d|$ of the determinant:

$$d \;=\; \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \ .$$

4. Equations for Planes

- The equation for a plane may be written in the form:

$$\vec{n} \cdot (x - x_0, y - y_0, z - z_0) \ = \ 0 \ ,$$

where $\vec{n}$ is a vector normal to the plane, and (x_0, y_0, z_0) is some arbitrary point in the plane. This formula simply expresses the fact that there is a vector (the normal direction) which is perpendicular to the displacement vector between any two points in the plane.

5. Intersections of Surfaces

- In general, if two surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$ in 3D intersect, they do so in a curve. To find a parametrization $\vec{r}(t)$ of this curve, one can proceed as follows: 1) Use the equations $F = 0$ and $G = 0$ to eliminate a variable, say z. 2) Use the new equation you just found in part 1 to write one variable as a function of the other, say $y = f(x)$. 3) Set the free variable to t, say $x = t$. Then $y(t) = f(x(t))$ from part 2, and one can then use either $F(x, y, z) = 0$ or $G(x, y, z) = 0$ to write z in terms of x, y, and thus in terms of t.

6. Length and Speed of Parameterized Curves

- The *speed* $s(t)$ along a parameterized curve $\vec{r}(t)$ is the length of the tangent vector $\vec{r}\,'(t)$. That is:

$$s(t) \ = \ \| \, \vec{r}\,'(t) \, \| \ .$$

- The *arc length* along a parameterized curve $\vec{r}(t)$ from $t = a$ to $t = b$ is simply the integral on $[a, b]$ of the speed, $length = \int_a^b s(t)dt$. In terms of the component functions of $\vec{r}(t)$ this formula reads:

$$length \ = \ \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} \ dt \ , \qquad \text{(2D)} \ ,$$

$$length \ = \ \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \ dt \ , \qquad \text{(3D)} \ .$$

7. Motion in 3D

- In three dimensional space we define the quantities:

$$\vec{r}(t) = \text{“position”} \ , \qquad \vec{v}(t) = \text{“velocity”} \ , \qquad \vec{a}(t) = \text{“acceleration”} \ .$$

These are *all vectors*, and they are related via the formulas:

$$\vec{v}(t) \ = \ \vec{r}\,'(t) \ , \qquad \vec{a}(t) \ = \ \vec{v}\,'(t) \ = \ \vec{r}\,''(t) \ .$$

- A basic principle is that given $\vec{a}(t)$ at *all* moments of time t, and given $\vec{r}_0 = \vec{r}(t_0)$ and $\vec{v}_0 = \vec{v}(t_0)$ at *one* moment of time $t = t_0$, then one can solve for both $\vec{v}(t)$ and $\vec{r}(t)$ at *all* moments of time t. One can do this by finding antiderivatives and setting initial conditions in the usual way.