FACT SHEET FOR 20C EXAM 2

Abstract. Here are some formulas that you should know for the second exam. Please bring a copy of this sheet to the test. Copies will **not** be included with the exams. Please do not write extra formulas on this sheet.

1. Tangent Planes and Linear Approximations

- For a function $z = f(x,y)$, the *tangent plane at* (x_0, y_0) is the best linear approximation to $f(x,y)$ for points (x,y) close to (x_0, y_0). It is given by the formula:

  $$L(x,y) \ = \ f(x_0,y_0) + \partial_x f(x_0,y_0)(x-x_0) + \partial_y f(x_0,y_0)(y-y_0) \ .$$

- Another way the think about the tangent plane is in terms of differentials:

  $$f(x+\Delta x, y+\Delta y) \ \approx \ f(x,y) + \partial_x f(x,y)\Delta x + \partial_y f(x,y)\Delta y \ .$$

  This last equation gives a good way to compute explicit approximations for f close to any point (x,y) where you know the true value (and the true value of the derivatives).

2. Gradients of Functions

- Recall that the gradient of a function $f(x,y)$ of two variables, or a function $g(x,y,z)$ of three variables, is the vector of all first partial derivatives:

  $$\nabla f \ = \ (f_x, f_y) \ , \qquad \nabla g \ = \ (g_x . g_y, g_z) \ .$$

- The gradient satisfies the following important rules (in 2D or 3D):

  1. If $\nabla f \neq (0,0)$ then it points in the *direction* of greatest increase.
  2. The length, $\| \nabla f \|$, gives the *rate* of greatest increase.
  3. If $\nabla f \neq (0,0)$, then it is orthogonal to the level sets $f = const.$ (at each fixed point).
  4. The gradient vanishes, $\nabla f(x,y) = (0,0)$, at a point (x,y) if and only if the tangent plane to $z = f(x,y)$ at (x,y) is parallel to the x-y plane (i.e. it is completely horizontal). In this case we call the point (x,y) where $\nabla f(x,y) = (0,0)$ a *critical point*.

- Given any unit vector $\vec{u}$, we form the *directional derivative* of $f(x,y)$ in the $\vec{u}$ direction according to the rule:

  $$D_{\vec{u}} f(x,y) \ = \ \vec{u} \cdot \nabla f(x,y) \ .$$

  This tells the rate of increase (or decrease if it is negative) of $f(x,y)$ in the $\vec{u}$ direction. It is **very important** that $\vec{u}$ is unit normalized to get this information. In general, if you are given any vector $\vec{v}$ representing the direction you care about, then you can turn it into a unit vector by dividing through by its length, i.e. by setting $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$.

3. CRITICAL POINTS; LOCAL MAX, MIN AND SADDLE POINTS

- Points where $\nabla f = (0,0)$ are called *critical points*. These are the *only* points where f can have a local max or local min. However, $\nabla f(x,y) = (0,0)$ **does not** imply that f has a local max or min at (x,y). It only gives a "necessary condition" for there to be a local max or min.
- At any critical point (x,y) we compute the *discriminant*:

  $$D(x,y) \;=\; f_{xx}(x,y)f_{yy}(x,y) - f_{xy}^2(x,y)\ .$$

  This gives us the following information (of course assuming that $\nabla f(x,y) = (0,0)$):

  (1) If $D(x,y) > 0$ then (x,y) is either a local max or min of f. In this case there are two possibilities:
      a) If $f_{xx}(x,y) > 0$ then f is *concave up* and (x,y) is a local min.
      b) If $f_{xx}(x,y) < 0$ then f is *concave down* and (x,y) is a local max.
  (2) If $D(x,y) < 0$ then (x,y) is a *saddle point* and (x,y) is **neither** a local max or min (in this case $f(x,y)$ has "mixed" behavior close to (x,y), meaning that it has both increasing and decreasing directions).
  (3) Finally, if $D(x,y) = 0$ then the second derivative test is inconclusive and we don't know anything without doing more work (we have not discussed this).

4. LAGRANGE MULTIPLIERS

- The method of *Lagrange multipliers* is a technique for finding local max and min of functions restricted to the boundary of a domain. The general setup is to give the boundary in terms of a constraint function $g(x,y)$. Then the problem says: *Try to find the local max and minima of $f(x,y)$ subject to the constraint $g(x,y) = c$.*

  For example, we could have $g(x,y) = x^2 + y^2$, in which case $g(x,y) = 1$ is a unit circle, and then we are simply trying to find local max and min of $f(x,y)$ for points (x,y) restricted to the unit circle.
- To solve the above problem, we set up the *Lagrange equations*:

  $$\nabla f(x,y) \;=\; \lambda \nabla g(x,y)\ , \quad g(x,y) \;=\; c\ .$$

  This turns out to be three equations in three unknowns, so in general the solution consists of finitely may points.
- To solve the Lagrange equations, it is sometimes helpful to use the following procedure:

  (1) First use the equation $\nabla f = \lambda \nabla g$ to eliminate λ in terms of a relation between x and y.
  (2) Substitute the relation on x and y from above into the constraint equation $g(x,y) = c$ to eliminate a variable, then solve for the remaining variable.

(3) Finally, once you have the values of one variable (either x or y) at the critical points, substitute these back into $g(x, y) = c$ to get the values of the other variable.

5. OPTIMIZATION PROBLEMS

- Using the methods of the previous two sections, it is possible to find the *global max* and *global min* of $f(x, y)$ on a finite domain Ω. The procedure is as follows:

  (1) First find all interior critical point where there could be an interior local max or min. Record all of these points, as well as the values of $f(x, y)$ at these points.
  (2) Second, find all critical points on the boundary by the method of Lagrange multipliers, or using 1D calculus if the boundary is a box. Record all of these points, as well as the values of $f(x, y)$ at these points.
  (3) Finally, compare all the values of $f(x, y)$ from 1)-2) above. The largest value is the global max, and smallest is the global min.

- It is important to keep in mind that the above procedure does not use the second derivative test. The second derivative test can only find *local* interior max or min, and you are better off just comparing the values of $f(x, y)$ at the (usually) finite number of critical points you find with the first derivative test and Lagrange's equation.