FACT SHEET FOR 20C FINAL EXAM

Abstract. Here are some formulas that you should know for the second exam. Please bring a copy of this sheet to the test. Copies will **not** be included with the exams. Please do not write extra formulas on this sheet.

1. Chain Rules and Implicit Differentiation

- (Chain Rule I) Given $f(x,y)$, $x(t)$, and $y(t)$, we can form a function of t alone by composing $h(t) = f\big(x(t), y(t)\big)$. Then:

$$\frac{dh}{dt} \;=\; \partial_x f \frac{dx}{dt} + \partial_y f \frac{dy}{dt} \;.$$

- (Chain Rule II) Given $f(x,y)$, $x(u,v)$, and $y(u,v)$, we can form a new function of (u,v) by composing $h(u,v) = f\big(x(u,v), y(u,v)\big)$. Then:

$$\begin{aligned} \frac{\partial h}{\partial u} &= \partial_x f \frac{\partial x}{\partial u} + \partial_y f \frac{\partial y}{\partial u} \;, \\ \frac{\partial h}{\partial v} &= \partial_x f \frac{\partial x}{\partial v} + \partial_y f \frac{\partial y}{\partial v} \;. \end{aligned}$$

- (Implicit Differentiation) Given a relation $F(x,y,z) = 0$, this defines an *implicit function* $z = f(x,y)$. One can then compute the partial derivatives of $z = f$ without solving for z via the formulas:

$$\frac{\partial z}{\partial x} \;=\; -\frac{F_x}{F_z} \;, \qquad \frac{\partial z}{\partial y} \;=\; -\frac{F_y}{F_z} \;.$$

2. Integrals Over Boxes

- (2D) In 2D we integrate a function $f(x,y)$ over the box $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$:

$$I \;=\; \int_a^b \int_c^d f(x,y)\,dy\,dx \;.$$

The integral is computed via one dimensional integrals evaluated in either order (x first or y first).

- (3D) In 3D we integrate a function $f(x,y,z)$ over the box $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$, $m \leqslant z \leqslant n$:

$$I \;=\; \int_a^b \int_c^d \int_m^n f(x,y,z)\,dz\,dy\,dx \;.$$

The integral is computed via one dimensional integrals evaluated in any order

3. Integrals Over 2D Simple Domains

- (Vertical Limits) We can integrate a function $f(x,y)$ over two dimensional domains $\mathcal{D} = \{a \leqslant x \leqslant b,\, \alpha(x) \leqslant y \leqslant \beta(x)\}$ by evaluating:
$$\iint_{\mathcal{D}} f(x,y)dxdy \;=\; \int_a^b \Big(\int_{\alpha(x)}^{\beta(x)} f(x,y)dy\Big)dx \;.$$
Here the order matters, and one must integrate y first.
- (Horizontal Limits) We can integrate a function $f(x,y)$ over two dimensional domains $\mathcal{D} = \{\alpha(y) \leqslant x \leqslant \beta(y),\, a \leqslant y \leqslant b\}$ by evaluating:
$$\iint_{\mathcal{D}} f(x,y)dxdy \;=\; \int_a^b \Big(\int_{\alpha(y)}^{\beta(y)} f(x,y)dx\Big)dy \;.$$
Here the order matters, and one must integrate x first.

4. Integrals Over 3D Cylindrical Domains

- We can integrate a function $f(x,y,z)$ over three dimensional domains $\mathcal{W} = \{(x,y) \in \mathcal{D}, \psi(x,y) \leqslant z \leqslant \phi(x,y)\}$ by evaluating:
$$I \;=\; \iiint_{\mathcal{W}} f(x,y,z)dxdydz \;=\; \iint_{\mathcal{D}} \Big(\int_{\psi(x,y)}^{\phi(x,y)} f(x,y,z)dz\Big)dxdy \;.$$
Here the order matters, and one must integrate z first.
- When setting up integrals of this form, it is often to follow a three step procedure:
  1. Find the range of the z variable. That is, first determine the bounds $\psi(x,y) \leqslant z \leqslant \phi(x,y)$. Sometimes this means computing one of $\psi(x,y)$ or $\phi(x,y)$ using geometry.
  2. Find the "projected" domain $\mathcal{D} \subseteq \mathbb{R}^2$. This will take the form of a simple domain like in 2D integral problems, and often this is found by intersection of the z limits from above (i.e. the boundary of $\mathcal{D}$ is given by the relation $\psi(x,y) = \phi(x,y)$).
  3. Plug all the information from 1) and 2) above to set up the triple integral I (remember to put z integration on the *inside*).

5. Integrals in Polar and Cylindrical Coordinates

- We may use polar coordinates:
$$x \;=\; r\cos(\theta) \;, \qquad y \;=\; r\sin(\theta) \;,$$
to write:
$$\iint_{\mathcal{D}} f(x,y)dxdy \;=\; \iint_{\mathcal{D}'} f\big(r\cos(\theta), r\sin(\theta)\big)rdrd\theta \;,$$
where $\mathcal{D}'$ is the polar description of $\mathcal{D}$.
- By using polar coordinates in (x,y), we can also transform 3D integrals as follows:
$$\iiint_{\mathcal{W}} f(x,y,z)dxdydz \;=\; \iiint_{\mathcal{W}'} f\big(r\cos(\theta), r\sin(\theta), z\big)rdrd\theta dz \;,$$
where $\mathcal{W}'$ is the cylindrical description of $\mathcal{W}$ (i.e. just changing the (x,y) variables over to polar, and leaving z fixed).