MATH 231A HOMEWORK 1: ELLIPTIC EQUATIONS

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Abstract. These homework problems are due in class on Monday, November 23d.

1. Weak solutions to general PDE

Show that for any function $H \in L^1_{loc}(\mathbb{R})$ that the function $u(x, y) = H(x - y)$ is a weak solution (in the sense of distributions) to the equation:

$$(\partial_x^2 - \partial_y^2)u \ = \ 0 \ ,$$

in every domain $\Omega \subset\subset \mathbb{R}^2$.

In particular, this shows that the notion of weak solutions to general non-elliptic second order PDE, even with C^∞ coefficients, really are "weak" in the sense that they are not functions with classically defined derivatives.

2. Localization of regularity for elliptic equations

For this problem, we'll agree to call a point $x_0 \in \Omega$ a "$C^{k,\alpha}$ point" for a function f if there exists a sufficiently small open neighborhood $B_r(x_0) \subset \Omega$ such that $f \in C^{k,\alpha}(B_r(x_0))$. Let:

$$(1) \quad L \ = \ \sum_{i,j} a_{ij}(x)\partial_i\partial_j + \sum_i b_i(x)\partial_i + c \ , \qquad a \in C^{k+1,\alpha}(\overline{\Omega}) \quad b, c \in C^{k,\alpha}(\overline{\Omega}) \ ,$$

be a uniformly elliptic equation with $C^{k,\alpha}$ coefficients. Prove that if $Lw = F$, then if x_0 is a $C^{k,\alpha}$ point for F we must have that x_0 is a $C^{k+2,\alpha}$ point for w. For simplicity, you may *assume* that $w \in C^{2,\gamma}$ for some $\gamma > 0$ to begin with.

As an example of this, suppose that (1) has C^∞ coefficients, and let $Lw = |x|^\alpha$ in the ball $\Omega = \{|x| < 1\}$. Then $w \in C^\infty(\{|x| < 1\} \setminus \{0\})$. One should note that this is in stark contrast to the situation of the previous problem where regularity of the RHS gives no additional regularity for the solution.

3. Subaverage value property

Let we say that a function $u \in L^1_{loc}$ is a "weakly subharmonic" function if the following inequality holds:

$$(2) \qquad \iint_\Omega u\Delta\varphi dx \ \geqslant \ 0 \ , \qquad \forall\varphi \in C_0^\infty(\Omega) \quad \text{with} \quad \varphi \geqslant 0 \ .$$

Prove that if u is weakly subharmonic in Ω, then for each strict subdomain $\Omega' \subset\subset \Omega$ that there exists a constant $C = C(\Omega', \Omega, u)$ (also depending on u!) such that $u \leqslant C$ a.e. in Ω', and furthermore that:

$$u(x) \ \leqslant \ \frac{1}{|B_r(x)|}\iint_{B_r(x)} u(y)dy \ , \qquad \text{for a.e.} \ \ x \in \Omega \quad \text{with} \quad B_r(x) \subseteq \Omega \ .$$

(Hint: Show that in each $\Omega' \subset\subset \Omega$ there exists a sequence of regularization $u^{(\epsilon)} \to u$ such that (2) is true for each $u^{(\epsilon)}$, thus reducing the problem to smooth subsolutions.)

4. Accuracy of Finite Elements

Recall that a weak $H^1_0(\Omega)$ solution to Poisson's equation $\Delta u = F$ is a function $w \in H^1_0(\Omega)$ such that:

$$\iint_\Omega \nabla u \cdot \nabla \varphi \, dx + \iint_\Omega F\varphi \, dx \ = \ 0 \ , \qquad \forall \ \ \varphi \in C^\infty_0(\Omega) \ ,$$

and also that such a solution is constructed via the minimization problem:

$$I[u] \ = \ \inf_{w \in H^1_0(\Omega)} I[w] \ , \qquad I[w] \ = \ \frac{1}{2}\iint_\Omega |\nabla w|^2 dx + \iint_\Omega Fw \, dx \ . \tag{3}$$

Recall the proof that such a solution exists is very close in spirit of finding minimizers to finite dimensional problems because we proceed via weak compactness. The point of this problem is to make this relationship even more explicit.

I) First, show that we can always find a minimizer $u^{(n)}$ to the associated family of *finite dimensional* variational problems:

$$I[u^{(n)}] \ = \ \inf_{w \in V_n} I[w] \ , \qquad I[w] \ = \ \frac{1}{2}\iint_\Omega |\nabla w|^2 dx + \iint_\Omega Fw \, dx \ , \tag{4}$$

where $V_n = \text{span}\{f_k\}_{k=1}^n$ is some finite dimensional subspace of $H^1_0(\Omega)$.

II) Next, show that one can uniformly estimate the difference between the solution $u^{(n)}$ to (4) and the true solution u to (3) by showing there exists a $C = C(\Omega)$, independent of n, such that:

$$\| \, u - u^{(n)} \, \|_{H^1(\Omega)} \ \leqslant \ C \inf_{w \in V_n} \| \, u - w \, \|_{H^1(\Omega)} \ . \tag{5}$$

Remark 4.1. The estimate (5) shows that the minimization (3) is remarkably stable. One can approximate the true solution via solutions in finite dimensional sub-spaces at the *same* level of accuracy that these sub-spaces can approximate u itself. In other words, if one picks V_n at a fairly good level of "resolution" for expressing reasonable functions (we assume the solution u is one of these), then V_n suffices for solving energy minimization problems at the same level of resolution. This is perhaps not too surprising because finding least squares approximations is just an "energy" minimization problem to begin with!

5. Existence of Minimizers for Dirichlet Energies With Bounded Coefficients

Recall that a matrix $a_{ij}(x)$ is said to be uniformly elliptic in Ω if there exists a $c = c(\Omega)$ with:

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \ \geqslant c|\xi|^2 \ , \qquad (x, \xi) \in \Omega \times \mathbb{R}^n \ .$$

I) Assuming only that $a_{ij} \in L^\infty(\Omega)$, show that the minimization problem of finding a $u \in H^1(\Omega)$ with $T_\Omega(u-g) = 0$ such that:

$$\sum_{i,j} \iint_\Omega a_{ij}(x)\partial_i u \partial_j u \ dx \ = \ \inf_{\substack{w\in H^1(\Omega) \\ T_\Omega w = T_\Omega g}} \sum_{i,j} \iint_\Omega a_{ij}(x)\partial_i w \partial_j w \ dx \ , \tag{6}$$

has a unique solution for each fixed $g \in H^1(\Omega)$ (the extension of a desired boundary value). (Hint: The proof here follows exactly the model case where $a_{ij} = \delta_{ij}$, the only difference being the proof of lower semicontinuity of the a_{ij} Dirichlet energy. To show this, try using the Riesz representation theorem.)

II) Show that the minimizer to (6) solves the weak equation:

$$\sum_{i,j} \iint_\Omega a_{ij}(x)\partial_i w \partial_j \varphi \ dx \ = \ 0 \ , \qquad \forall \ \ \varphi \in C_0^\infty(\Omega) \ .$$

6. Harmonic Maps into spheres I

In this problem, we consider vector-valued functions $\Phi, \Psi : \mathbb{R}^3 \to \mathbb{R}^3$. A problem that comes up often in physics and geometry is to look at minimizers to the constrained variational problem of finding a Φ such that:

$$\iint_\Omega |\nabla \Phi|^2 \ dx \ = \ \inf_{\substack{\Psi \in H^1(\Omega), |\Psi|=1 \\ T_\Omega \Psi = \Phi_0}} \iint_\Omega |\nabla \Psi|^2 \ dx \ . \tag{7}$$

That is, one minimizes the usual Dirichlet energy subject to the constraint $\Psi \in \mathbb{S}^2$ for all competing functions. This restriction is *non-linear*, and so it produces a non-linear version of Laplace's equation for the Euler-Lagrange equations of the corresponding minimizer.

I) First of all, show that the Euler-Lagrange equations for a critical point (say local minimum) to the problem (7) solves (weakly) the system of equations:

$$\Delta \Phi \ = \ -\Phi |\nabla \Phi|^2 \ . \tag{8}$$

(Hint: If you are good with Lagrange multipliers this is probably not too difficult. Otherwise, look at variations of the form $\Phi_t = \frac{\Phi + t\varphi}{|\Phi + t\varphi|}$ where $\varphi \in C_0^\infty(\Omega)$.)

II) Second, show that bounded weak solutions to the equation (8) *do not* have to be continuous (even though the coefficients are constant!). For example, show that the function:

$$\Phi(x) \ = \ \frac{x}{|x|} \ , \tag{9}$$

is a weak solution to (8) in the sense that $\Phi \in H^1(\Omega)$ and:

$$\iint_\Omega \nabla \Phi \cdot \nabla \vec{\varphi} \ dx - \iint_\Omega \vec{\varphi} \cdot \Phi |\nabla \Phi|^2 \ dx \ = \ 0 \ , \quad \forall \ \ \vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3) \in [C_0^\infty(\Omega)]^3 \ .$$

Remark 6.1. Notice that the solution (9) is such that $\Phi|_{|x|=1} = Id$, i.e. its boundary values are C^∞. This loss of interior regularity is a typical phenomena for solutions to non-linear systems, even ones that show up in natural energy minimization problems. Basically, what is happening here is that the non-linear constraint $\Phi \in \mathbb{S}^2$ leads to a lack of global convexity for the minimization problem, which tends to cause a lot of trouble when proving estimates.

7. Project A - Compactness of smooth harmonic maps into hyperbolic space $\mathbb{H}^2$

In this problem, we are looking at compactness properties of classical solutions to the Euler-Lagrange equations of the following non-linear energy for complex valued functions:

$$z : \Omega \subset \mathbb{R}^3 \ \rightarrow \ D \subset \mathbb{C} \ ,$$

where $D = \{z \in \mathbb{C} \,\big|\, |z| < 1\}$ is the unit disk:

$$E[z] \ = \ \frac{1}{2}\iint_\Omega (1 - |z|^2)^{-2}|\nabla z|^2 dx \ . \tag{10}$$

Notice that the is the Dirichlet energy corresponding to the Poincare metric on D (i.e. ∇z is a tangent vector over D, and we use the metric $ds^2 = \frac{dzd\overline{z}}{(1-|z|^2)^2}$ to take its inner product as opposed to the usual euclidean metric on D, which would give linear harmonic functions instead).

I) First, show that the Euler-Lagrange equations for the functional (10) are:

$$\Delta z \ = \ -\frac{2\overline{z}}{(1-|z|^2)}\nabla z \cdot \nabla z \ , \qquad |z| < 1 \ . \tag{11}$$

II) Next show that if z is a classical (smooth) solution to (11) that its Dirichlet energy density satisfies the "Böchner identity":

$$\frac{1}{2}\Delta\Big[(1-|z|^2)^{-2}|\nabla z|^2\Big] \ = \ (1-|z|^2)^{-2}\sum_{i,j}|D_i\partial_j z|^2 + 2(1-|z|^2)^{-4}\Big[\sum_{i,j}|\partial_i z\overline{\partial_j z}|^2 - |\sum_i(\partial_i z)^2|^2\Big] \ , \tag{12}$$

where we are using the shorthand $D_j = \partial_j + \frac{2\overline{z}}{1-|z|^2}\partial_j z$. (Hints: You may find proving this formula to be quite messy, but notice that one has the following identities:

$$D_i\partial_j z \ = \ D_j\partial_i z \ ,$$
$$\partial_i\langle X, Y\rangle \ = \ \langle D_i X, Y\rangle + \langle X, D_i Y\rangle \ ,$$
$$(D_iD_j - D_jD_i)X \ = \ -4\frac{1}{(1-|z|^2)^2}\Im(\partial_i z\overline{\partial_j z})\, X \ ,$$

where X, Y are any two complex valued functions, and we are writing:

$$\langle X, Y\rangle \ = \ \frac{1}{(1-|z|^2)^2}X\overline{Y} \ .$$

Finally, notice that the equation (11) can be written in the form:

$$\sum_i D_i\partial_i z \ = \ 0 \ .$$

These identities express the fact that D_i is a torsion free metric connection on a complex line bundle over $\mathbb{R}^3$ with curvature induced by the Poincare metric.)

III) Next, show that if $z^{(n)}$ is a sequence of "bounded" solutions to (11) with uniformly bounded energy, meaning that for each there exists a universal constants $C > 0$ and $0 < c < 1$ with:

$$E[z^{(n)}] \ \leqslant \ C \ ,$$

and:

$$|z^{(n)}(x)| \ \leqslant \ c \ , \quad \text{in} \ \ \Omega \ ,$$

then there exists a subsequence $z^{(n_i)}$ and a smooth solution $z^{(\infty)}$ to (11) with:

$$\| \, z^{(n_i)} - z^{(\infty)} \, \|_{C^k(\Omega')} \ \to \ 0 \ ,$$

for all k and subdomains $\Omega' \subset\subset \Omega$. (Hint: Use the Böchner identity (12) and Problem 3 to produce an a-priori gradient bound of the form:

$$|\nabla z^{(n)}| \ \leqslant \ M \ , \quad \text{in} \ \ \Omega' \ ,$$

and then apply Schauder theory and some convergence theorems you know from analysis.)

IV) Finally, show that the above compactness property is definitely false for solutions to (8) by considering rescalings $\Phi_\lambda(x) = \Phi_1(\lambda x)$ of the inverse stereographic projection:

$$\Phi_1(x^1, x^2, x^3) \ = \ \Big(\frac{2x^1}{1 + (x^1)^2 + (x^2)^2}, \frac{2x^2}{1 + (x^1)^2 + (x^2)^2}, \frac{-1 + (x^1)^2 + (x^2)^2}{1 + (x^1)^2 + (x^2)^2} \Big) \ .$$

Remark 7.1. This problem shows that for some non-linear elliptic systems there are good compactness properties, while for others there is a loss of compactness, even when comparing equations that are in the same "family" (e.g. harmonic maps). There is a simple geometric reason why solutions to (11) are so well behaved, which is that the underlying curvature of the target (i.e. the disk D) is *negative*. This leads to a system of equations with good convexity properties. Notice that the target curvature in the case of (8) was positive. There is also an identity of the form (12) for (8), but it has a negative sign for the second term due to the target curvature, so it is not coercive (e.g. the non-compact sequence above causes the RHS to go to $-\infty$).

Problem 8: Project B - Full regularity of critical points to scalar variational problems with continuous gradients

In this problem, we consider real valued functions:

$$u : \Omega \subset \mathbb{R}^3 \ \to \ \mathbb{R} \ ,$$

which are critical points of the following variational problem:

$$E[u] \ = \ \iint_\Omega F(\nabla u) \ dx \ , \qquad \sum_{i,j} \partial_i \partial_j F(\eta) \xi_i \xi_j \geqslant c(\eta) |\xi|^2 \ . \tag{13}$$

A basic example of this is the minimal surface density $F(\eta) = \sqrt{1 + |\eta|^2}$. Recall that the Euler-Lagrange equations to (13) are:

$$\sum_i \partial_i \big[F_{\eta^i}(\nabla u) \big] \ = \ 0 \ . \tag{14}$$

In this problem, you will show that if we a-priori assume that ∇u is *continuous*, then if it is a weak solution to (14) it is in $C^\infty(\Omega)$. That is, continuity of the gradient for a critical point to (13) implies complete regularity.

To show this, we'll reduce the problem in a series of steps, and you may work in strict domains $\Omega' \subset\subset \Omega$ to avoid distracting boundary regularity issues:

I) Show first of all that if u a weak solution to (14) with bounded gradient, then in fact $\nabla^2 u \in L^2(\Omega')$, and $\nabla u = w$ is a weak $H^1(\Omega')$ solution to the divergence form system of equations:

$$\sum_{i,j} \partial_i \big[F_{\eta^i \eta^j}(w) \partial_j w \big] \ = \ 0 \ . \tag{15}$$

Notice that the assumption on $\nabla^2 F$ on line (13) and the continuity of w implies that this is a uniformly elliptic equation. (Hint: Formally speaking, one can see immediately that (15) is the correct from by simply differentiating equation (11). Again, the problem here is making sense of these kind of higher derivative computations for weak functions. Use difference-quotients.)

II) The next step is the most crucial one. Here we'll show that the solution w to (15) is in fact in $C^\alpha(\Omega')$, for any $\alpha \in (0,1)$. Notice that if we *already knew* that $w \in C^\alpha(\Omega')$, then we could get higher regularity through the usual divergence form Schauder estimate. The problem here is that we only know that the coefficient matrix $F_{\eta^i \eta^j}(w) \in C^0(\Omega')$. This is of course a general problem: Show that if w is a solution to the weak equation:

$$\sum_{i,j} \iint_{\Omega'} a_{ij}(x) \partial_i w \partial_j \varphi \ = \ 0 \ , \qquad \forall \ \ \varphi \in C_0^\infty(\Omega') \ ,$$

where $a_{ij}(x)$ is a continuous and uniformly elliptic matrix, then in fact $w \in C^\alpha(\Omega')$ for all $\alpha \in (0,1)$. (Hint: Try to proceed via business as usual. First, use the harmonic approximation lemma, and then try to reduce things to a modified iteration lemma for monotonic quantities $I(\rho)$ which obey an inequality of the form:

$$I(\rho) \ \leqslant \ C\Big[\Big(\frac{\rho}{r}\Big)^n I(r) + \epsilon(r) I(r) \Big] \ ,$$

where $\epsilon(r) = o(1)$ as $r \to 0$, and C is a uniform constant independent of ρ and r.)

III) Finally, show that one can bootstrap $C^\alpha(\Omega')$ regularity for solutions of (15) to full $C^\infty(\Omega')$ regularity.

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