INTRODUCTION TO THE LAPLACE EQUATION

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Abstract. These are companion lecture notes for Math 231A on Monday 9/28/09.

1. A "Derivation" of the Laplace Equation

As explained in class, one should think of solutions to the Laplace equation $\Delta u = 0$ as infinite time solutions to the heat equation $\partial_t = k\Delta u$ (e.g. as "steady state" temperature distributions). To understand this better, consider the Dirichlet boundary value problem for a heat equation in some finite domain $\Omega \subset\subset \mathbb{R}^n$:

$$\partial_t u \ = \ k\Delta u \ , \qquad \text{in} \ \ [0,\infty)\times\Omega \ , \tag{1a}$$

$$u|_{\partial\Omega} \ = \ g \ . \tag{1b}$$

There are many solutions to this problem, although each one is uniquely determined by its Cauchy data $u(0,x) = u_0(x)$. For now we are not interested in the problem close to $t = 0$, but rather the asymptotic problem $t \to \infty$.

Perhaps the most direct way to understand solutions to (1) is to split them into two pieces:

$$u(t,x) \ = \ w(t,x) + u_\infty(x) \ ,$$

where:

$$\partial_t w \ = \ k\Delta w \ , \qquad \text{in} \ \ [0,\infty)\times\Omega \ ,$$

$$w|_{\partial\Omega} \ = \ 0 \ ,$$

represents the "local in time fluctuations" of $u(t,x)$, and:

$$\Delta u_\infty \ = \ 0 \ , \qquad \text{in} \ \ \Omega \ , \tag{2a}$$

$$u_\infty|_{\partial\Omega} \ = \ g \ , \tag{2b}$$

which represents the "asymptotic state" of $u(t,x)$. Given any solution $u(t,x)$ to (1) one can always enforce this splitting by first *defining* u_∞ according to equation (2) and then setting $w = u - u_\infty$ (it turns out there are always solutions to problem (2) for reasonable boundary data $g(x)$, as will be shown later). We claim that $\lim_{t\to\infty} u(t,x) = u_\infty(x)$, which justifies the choice of this splitting. That is, u_∞ is the final state of the time dependent solution to (1). In an average sense, this convergence is given succinctly by the following:

Proposition 1.1 (Convergence theorem for the zero Dirichlet heat flow)**.** *Let $w(t,x)$ be a C^2 solution to (1) with $g \equiv 0$. Then one has the following estimate:*

$$\| \, w(t) \, \|_{L^2(\Omega)} \ \leqslant \ e^{-\lambda_{\Omega,k} t} \| \, w(0) \, \|_{L^2(\Omega)} \ , \tag{3}$$

where $\lambda_{\Omega,k} > 0$ is a universal constant depending only on Ω and k.

In particular, this shows that the convergence of $u \to u_\infty$ is exponentially fast. For those who are familiar with eigenfunction expansions and separation of variables should be able to prove this without too much trouble. Here is another proof that uses only the equation and integration by parts:

Proof of estimate (3). We start by estimating (assume WLOG that w is real valued):

$$\iint_\Omega w^2(t,x)dx \ \leqslant\ C_\Omega \iint |\nabla_x w(t,x)|^2 dx \ , \qquad \text{if} \ \ w|_{\partial\Omega} = 0 \ ,$$

where C_Ω is some constant depending only on Ω. This is known as "Poincare's inequality", and there is a simple proof of it in Section 5.8.1 of Evans (actually the proof if Evans is for a function minus its average, but the same idea works for a function with zero boundary value if you work with "traces"; we may get to more details on these kinds of estimates later). The important thing is that for an inequality like this to work, we *must* have $w|_{\partial\Omega} = 0$ because one can see that $w \equiv const. \neq 0$ would provide a counterexample otherwise.

Using the above inequality and integration by parts (i.e. Green's theorem which also uses that $w|_{\partial\Omega} = 0$) we then compute:

$$\begin{aligned}
\iint_\Omega w^2(t,x)dx \ &\leqslant\ C_\Omega \iint |\nabla_x w(t,x)|^2 dx \ , \\
&=\ -C_\Omega \iint w(t,x)\Delta w(t,x)\, dx \ , \\
&=\ -\frac{C_\Omega}{k} \iint w(t,x)\partial_t w(t,x)\, dx \ , \qquad \text{using} \ \ \partial_t w = k\Delta w \ , \\
&=\ -\frac{C_\Omega}{2k}\frac{d}{dt} \iint w^2(t,x)\, dx \ .
\end{aligned}$$

Rearranging this inequality, we have proved that if w solves (1) with $g \equiv 0$ then:

$$\frac{d}{dt}\ln\big(\| w(t) \|_{L^2(\Omega)}\big) \ \leqslant\ -\frac{k}{C_\Omega} \ .$$

Setting $\lambda_{\Omega,k} = \frac{k}{C_\Omega}$ and integrating from time $s = 0$ to time $s = t$ we have:

$$\ln\big(\| w(t) \|_{L^2(\Omega)}\big) \leqslant -\lambda_{\Omega,k} t + \ln\big(\| w(0) \|_{L^2(\Omega)}\big) \ .$$

Finally, we arrive at (3) by exponentiating both sides of this last line. □

Notice that as a byproduct of the proof, one can imagine from physical considerations that the Poincare constant C_Ω should go to ∞ as the size of Ω grows. That is, $\lambda_{\Omega,k} \to 0$ as Ω becomes larger and larger. The "physical explanation" for this is that it should a take freezing boundary temperature longer to effect the middle of Ω for larger domains.

2. The 2D Laplacian and Complex Analysis

There is a very special relationship between the $n = 2$ case of the Laplace equation and the Cauchy-Riemann equations. Recall that in 2D we have $\Delta = \partial_x^2 + \partial_y^2$, so by simply factoring we can write:

$$\Delta \ =\ \partial\overline{\partial} \ , \qquad \text{where} \ \ \partial \ =\ \partial_x - i\partial_y \ , \ \ \overline{\partial} \ =\ \partial_x + i\partial_y \ . \tag{4}$$

In particular, any solution to either of the equations $\overline{\partial}u = 0$ or $\partial u = 0$ is automatically a solution to $\Delta u = 0$, i.e. a harmonic function. That is to say, both holomorphic (complex analytic) functions and anti-holomorphic functions (their complex conjugates) are solutions to Laplace's equation. What is even more interesting is that there is a converse to this, which is contained in the following basic result:

Proposition 2.1 (Structure of 2D harmonic functions)**.** *Let $\overline{\partial} u = 0$ in some $\Omega \subseteq \mathbb{R}^2 \cong \mathbb{C}$. Then both $\Re(u)$ and $\Im(u)$ are harmonic functions. That is, if $u = f + ig$ with f, g real, then $\Delta f = \Delta g = 0$.*

Conversely, if f is any real valued harmonic function in a simply connected domain $\Omega \subseteq \mathbb{R}^2 \cong \mathbb{C}$, there exists a real harmonic function g with the property that $u = f + ig$ is holomorphic. In particular, every (complex valued) harmonic function w is a sum of a holomorphic and an anti-holomorphic function:

$$w(x, y) \ = \ \sum_{n=0}^{\infty} a_n (x + iy)^n + \sum_{n=0}^{\infty} b_n (x - iy)^n \ . \tag{5}$$

Proof. The proof of the first part is a direct consequence of the splitting (4). If $\overline{\partial} u = 0$ then $\Delta u = 0$, and since Δ has no complex coefficients it respects the real and imaginary parts of u.

To prove the converse, suppose we are given the real part f with $\Delta f = 0$. Then by the Cauchy-Riemann equations we are trying to find a real g that solves the system:

$$\partial_x g \ = \ -\partial_y f \ := \ \omega_1 \ , \qquad \partial_y g \ = \ \partial_x f \ := \ \omega_2 \ .$$

Given (ω_1, ω_2) the above system is in general overdetermined and has no solution. However, if the compatibility condition $\partial_y \omega_1 = \partial_x \omega_2$ is satisfied, then there is a unique solution modulo constants in any simply connected domain (this is the "Poincare lemma", which is a standard 20E or "calculus on manifolds" result). In the present case, one can see that the compatibility condition is equivalent to $\Delta f = 0$, so we can find our g.

Finally, the formula (5) follows at once from the above considerations and the power series expansion for holomorphic and anti-holomorphic functions. □

Notice that one artifact of the proof is that *any harmonic function u must in fact be real analytic*. In particular a harmonic function u it is automatically in $C^\infty(\Omega)$, which is a very strong regularity result (e.g. compare again to the situation of a transport equation in 2D as in the previous notes).

2.1. **Harmonic polynomials.** As a quick application of the last Proposition, its not too hard to come up with a procedure to generate all (say real) harmonic polynomials of a given degree. In fact, for each degree $d \geqslant 1$ there are exactly 2 of them, and they are given by the real and imaginary parts of $(x + iy)^d$. For example, a basis for these polynomials up to degree $d = 3$ is:

$$\begin{aligned}
d &= 0 & & 1 \ , \\
d &= 1 & & x \ , \ y \ , \\
d &= 2 & & x^2 - y^2 \ , \ xy \ , \\
d &= 3 & & x^3 - 3xy^2 \ , \ y^3 - 3x^2 y \ .
\end{aligned}$$

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