CONTRAST OF A LINEAR HYPERBOLIC AND ELLIPTIC FIRST ORDER PDE

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ABSTRACT. These are companion lecture notes for Math 231A on Friday 9/25/09.

1. INTRODUCTION

These notes illustrate a typical difference between solutions to "elliptic" and "hyperbolic" PDE. In the former case one has strong smoothness and compactness properties, while in the later there is lack of smoothing and obvious compactness properties (this is not to say there there are *no* compactness properties for some hyperbolic PDE, but they are a bit more subtle and less useful).

We start with solutions to the first order PDE:

$$\overline{\partial} u \ = \ 0 \ , \qquad \overline{\partial} \ = \ \partial_x + i\partial_y \ . \tag{1}$$

Here $u : \Omega \subseteq \mathbb{R}^2 \to \mathbb{C}$ is some C^1 function. Writing u in terms of its real and imaginary parts $u = f + ig$ we can unwind (1) as being the Cauchy-Riemann system:

$$\partial_x f \ = \ \partial_y g \ , \qquad \partial_y f \ = \ -\partial_x g \ .$$

Therefore, u must in fact be complex analytic:

$$u(x,y) \ = \ \sum_n^\infty a_n (x+iy)^n \ .$$

In particular, it must be a $C^\infty(\Omega)$ function.

Next, consider the simple transport equation:

$$(\partial_x + \partial_y) w \ = \ 0 \ . \tag{2}$$

Here again we can take $w : \Omega \subseteq \mathbb{R}^2 \to \mathbb{C}$ to be some C^1 function (just so the equation makes sense classically). An direct calculation shows that any function of the form:

$$w(x,y) \ = \ F(x-y) \ ,$$

where F is an arbitrary C^1 function, solves (2). In particular, solutions to (2) are in general *only* C^1 functions. This is in sharp contrast to the situation for (1) above.

2. COMPACTNESS

To further illustrate the differences between (1) and (2) we look at a notion of compactness in these contexts.

If u is a solution to (1), then the one-from $\omega(z,\zeta) = \frac{u}{z-\zeta}dz$ is closed ($d\omega = 0$) away from the singularity $z = \zeta$. Thus, by the Stokes theorem and a little bit of careful calculation one has the integral formula:

$$u(\zeta) \ = \ \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{u(z)}{z-\zeta} dz \ , \tag{3}$$

where $\mathcal{C} \subseteq \Omega$ is any contour inside the domain where (1) holds.

Next, let $u^{(n)}$ be a sequence of solutions to (1). Suppose further that this sequence is uniformly bounded inside of Ω:

$$\| u^{(n)} \|_{L^\infty(\Omega)} \ \leqslant \ C \ . \tag{4}$$

Then by applying equation (3) to differences of each of the $u^{(n)}$ evaluated at various points $\zeta, \zeta' \in \Omega$ we have the *uniform* family of Lipschitz bounds:

$$|u^{(n)}(\zeta) - u^{(n)}(\zeta')| \ \leqslant \ CA_{\mathcal{O}}|\zeta - \zeta'| \ ,$$

for any fixed subdoman $\mathcal{O} \subset\subset \Omega$, where $A_{\mathcal{O}}$ is some additional constant (basically depending on how close $\partial\mathcal{O}$ is to $\partial\Omega$). In particular, the sequence $u^{(n)}$ is equicontinuous, and by the Arzelà-Ascoli theorem there is a subsequence $u^{(n_k)}$ that converges uniformly on compact sets to some u_0. Therefore, one has strong convergence of this subsequence in all $L^p(\mathcal{O})$ spaces.

Now, compare this situation to that of solutions to (2). Notice that the sequence of functions:

$$w^{(n)}(x,y) \ = \ e^{in(x-y)} \ ,$$

all solve (2), are even C^∞, and obey a uniform bound of the form (4) with $C = 1$. However, a standard fact from real analysis (the Riemann-Lebesgue lemma) is that $w^{(n)} \rightharpoonup 0$ in any $L^p(\mathcal{O})$ space. However, it is easy to see that they also have the uniform L^p bound:

$$\| w^{(n)} \|_{L^p(\mathcal{O})} \ = \ |\mathcal{O}|^{\frac{1}{p}} \ .$$

In particular, any subsequence $w^{(n_k)}$ cannot converge strongly in *any* $L^p(\mathcal{O})$ space because if this was the case one must have norm convergence to $w_0 = 0$, which violates the equality on the last line above.

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