NOTES ON DIRECT METHODS TO SOLVE THE DIRICHLET PROBLEM

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Abstract. These are companion lecture notes for Math 231A on Friday 10/2/09.

1. Introduction

In these notes we'll discuss some aspects of trying to solve the problem:

$$\Delta u = 0 , \qquad \text{in } \Omega , \tag{1a}$$

$$u|_{\partial\Omega} = g , \tag{1b}$$

where $g \in C^2(\partial\Omega)$ is given. Recall that according to the "Dirichlet principle", a $C^2(\overline{\Omega})$ solution to the problem (1) is equivalent to a solution of the problem:

$$\text{Find } u \in C^2(\overline{\Omega}) \text{ with } u|_{\partial\Omega} = g \text{ and such that:} \quad E[u] = \inf_{w \in C^2(\overline{\Omega})} E[w] , \tag{2}$$

where E is the usual Dirichlet energy:

$$E[u] = \frac{1}{2}\iint_{\Omega} |\nabla u|^2 dx . \tag{3}$$

In other words, minimize the energy functional $E : C^2(\overline{\Omega}) \to \mathbb{R}$ on the (convex) subset of functions $w \in C^2(\overline{\Omega})$ with $w|_{\partial\Omega} = g$.

Ideally, one would like to approach this problem in the same way that one approaches problems in the analysis of functions on $\mathbb{R}^n$. For example, recall the basic result that *any continuous function* $f : \mathbb{R}^n \to \mathbb{R}$ *takes an absolute minimum on any compact set* $K \subseteq \mathbb{R}^n$. It is easy to see that the function (3) is continuous on the subset of $C^2(\overline{\Omega})$ with boundary value g, but the problem is that the set:

$$S_C = \{ w \in C^2(\overline{\Omega}) \,\big|\, w|_{\partial\Omega} = g, E[w] \leqslant C \} ,$$

is not compact for any $E_0 < C$, where E_0 is the minimum energy for $w|_{\partial\Omega} = g$ (see if you can prove this!). Thus, it is not entirely clear if the minimum can actually be attained.

In general, the use of compactness is a much more subtle in infinite dimensional spaces than it is in finite dimensions. Roughly speaking, the situation is as follows: On an infinite dimensional normed vector space, a closed and bounded subset is almost never compact in the norm topology. To gain compactness, what one needs to do is to work with *two* norms, the first being the strong norm that defines closed balls to begin with. Then one weakens this topology so that there are many less open sets. In this weaker topology, it becomes much easier to select finite subcovers, so one gets compactness of the balls coming from the original norm topology.

The implementation of the above idea is a bit trickier than it looks at first. For example, the space $C^3(\overline{\Omega}) \subset C^2(\overline{\Omega})$ compactly embeds balls by the Arzelà-Ascoli theorem, so one could hope that this gives a weak/strong topology pair that works to provide minimums for continuous (say even convex) functionals. However, the following example shows that this is not the case.

1.1. A convex functional with no C^2 minimum.

Consider the functional:

$$I[u] \ = \ \int_{-1}^{1} x^2 u_x^2 dx \ .$$

We claim that:

$$\inf_{w\in S} I[w] \ = \ 0 \ , \qquad S = \{w \in C^2([-1,1]) \ \big| \ w(-1) = -1, w(1) = 1\} \ . \tag{4}$$

Clearly there is no $u \in C^2([-1,1])$ with nonzero boundary conditions and such that $I[u] = 0$, which provides the counterexample.

To see that (4) holds, consider the sequence of functions $w^{(n)} = \varphi(nx)$ where φ is any monotonic odd function with the property:

$$\varphi(x) \equiv \ -1 \ , x \leqslant -\frac{1}{2} \ , \quad \varphi(x) \equiv \ 1 \ , x \geqslant \frac{1}{2} \ .$$

An easy calculation then shows:

$$x^2 (w_x^{(n)})^2 \ \leqslant C_\varphi \ , \qquad x^2 (w_x^{(n)})^2 \ \equiv \ 0 \text{ for } |x| \geqslant \frac{1}{2n} \ .$$

In particular $I[w^{(n)}] = O(\frac{1}{n})$.

Notice that the problem with this example is that its a lot like trying to minimize the function $f(x) = 1 - \tanh(x)$ on all of $\mathbb{R}$. One can take a sequence of points $x \to \infty$ where $f \to 0$, but you never hit the minimum. Notice that the sequence $w^{(n)}$ constructed above becomes very large in $C^2([-1,1])$, but it does not make $I[w^{(n)}]$ large.

Based on our experience now, and still following the "strong/weak" topology philosophy explained above, we see that to minimize (3) we are looking for a norm N and a topology W with the following properties:

- The norm N of $w^{(n)}$ must not go to infinity if $E[w^{(n)}] \leqslant C$.
- The balls in N should be compact in W.
- The functional E should be as "continuous as possible" with respect to convergence in W.

The last requirement above is an additional important technicality. Since we only expect to get convergence of our minimizing sequence with respect to W, we are not automatically guaranteed that the energy of the limit doesn't "jump" up to some value larger than the greatest lower bound E_0. This consideration is not entirely trivial, because as will ne seen shortly, E will *not* be continuous with respect to the W topology.

2. THE SPACES $H^1(\Omega)$, $H^1_0(\Omega)$, AND TRACES

For details on Sobolev spaces, please refer to Evans Sections 5.2-5.3 and 5.5. For starters, we recall:

Proposition 2.1 (Structure of $H^1(\Omega)$ and $H^1_0(\Omega)$)**.** *For test functions* $\varphi \in C^\infty(\overline{\Omega})$ *define the norm:*

$$\| \varphi \|^2_{H^1(\Omega)} \ = \ \iint_\Omega |\nabla\varphi|^2 dx + \iint_\Omega |\varphi|^2 dx \ . \tag{5}$$

Then the space $H^1(\Omega)$ *which is the completion of* $C^\infty(\overline{\Omega})$ *with respect to* $\| \cdot \|_{H^1(\Omega)}$ *is a Hilbert space. Furthermore, this space can be identified as the subset of* $L^2(\Omega)$ *functions* u *with the property that* $\nabla u \in L^2(\Omega)$ *in the sense of distributions.*

If one defines $H^1_0(\Omega)$ *as the completion of* $C^\infty_0(\Omega)$ *(test functions vanishing on* $\partial\Omega$*) with respect to* (5)*, then this is also a Hilbert space, and the inclusion* $H^1_0(\Omega) \subset H^1(\Omega)$ *is closed.*

Proof. The norm $\| \cdot \|^2_{H^1(\Omega)}$ is clearly given by an inner product, and the space is complete by definition. The only interesting thing is the identification with weakly differentiable functions. This is in Evans Chapter 5.3, and boils down to the standard theorems which say that everything is approximated well by smooth functions. □

2.1. **Traces.** To complete the picture, we need a similar identification of $H^1_0(\Omega)$ with weakly differentiable functions. The issue is that we need to make sense of the statement "u restricted to $\partial\Omega$ vanishes". This is non-trivial because $u \in H^1(\Omega)$ only defined almost everywhere so in general there in no way to make sense of it on a set of measure zero. However, there is the following:

Proposition 2.2 (Restriction Theorem)**.** *Let* $u \in H^1(\Omega)$*. Then there exists a function* $g \in L^2(\partial\Omega)$ *such that if* $u^{(n)} \in C^\infty(\overline{\Omega})$ *is any sequence with* $u^{(n)} \to u$ *in* $H^1(\Omega)$*, we also have* $u^{(n)}|_{\partial\Omega} \to g$ *in* $L^2(\partial\Omega)$*. Furthermore, we have the bound:*

$$\| g \|_{L^2(\partial\Omega)} \ \leqslant \ C_\Omega \| u \|_{H^1(\Omega)} \ .$$

Proof. This is a special case of Theorem 1 in Chapter 5.5 of Evans. □

Using the above proposition, we can define a boundary value $u|_{\partial\Omega}$ for any $u \in H^1(\Omega)$ by simply taking the limit of the approximating boundary values. We call this the "trace", and define the corresponding linear operator $T_\Omega u = u|_{\partial\Omega}$. As a special case of this, notice that $H^1_0(\Omega)$ is automatically identified with the functions $u \in H^1(\Omega)$ such that $T_\Omega u = 0$.

3. The weak formulation of the problem, and first estimates

As one might guess, we will look for solutions to (2) is the space $H^1(\Omega)$. This seems natural given that the energy functional is sort of like the $H^1(\Omega)$ norm to begin with. However, the price one pays is that the corresponding solution is a-priori only in $H^1(\Omega)$, and not the desired $C^2(\overline{\Omega})$ function. There is no way to fix this issue in the context of the existence theorem, and one treats the smoothness of the weak solutions we find here as a different problem (i.e. using a more involved set of techniques).

The first thing we will do is to reduce the problem to an auxiliary one: Given a function $F \in L^2(\Omega)$, find a function $v \in H^1_0(\Omega)$ which solves the problem:

$$I[v] \ = \ \inf_{w \in H^1_0(\Omega)} I[w] \ , \qquad I[w] \ = \ \frac{1}{2}\iint_\Omega |\nabla w|^2 dx + \iint_\Omega F w \, dx \ . \tag{6}$$

It is not hard to see that if we are given $g \in C^2(\overline{\Omega})$ (a C^2 extension of the boundary value from (1b)), and if we can find v as above, then the function $u = v + g$ minimizes (2) if we have set $F = -\Delta g$ in (6), because clearly:

$$\inf_{w \in H^1(\Omega), T_\Omega w = g} E[w] \ = \ \inf_{w \in H^1_0(\Omega)} E[w+g] \ = \ \inf_{w \in H^1_0(\Omega)} I[w] + \frac{1}{2}\iint_\Omega |\nabla g|^2 dx \ .$$

Notice also that by the same calculation used in the derivation of the Dirichlet principle, one has that a solution to the minimization problem (6) solves:

$$\iint_\Omega \nabla u \cdot \nabla \varphi \, dx + \iint_\Omega F \varphi \, dx \ = \ 0 \ , \quad \text{for all } \varphi \in C^\infty_0(\Omega) \ .$$

This is sometimes called the "weak formulation" of the zero Dirichlet data Poisson problem:

$$\begin{aligned} \Delta u \ &= \ F \ , \qquad \text{in } \ \Omega \ , \\ T_\Omega u \ &= \ 0 \ . \end{aligned}$$

For the remainder of the lecture, we focus on trying to solve (6).

3.1. **A preliminary estimate.** Recall that from the introduction, we need to make sure that $I[w^{(n)}]$ goes to infinity if $\| w^{(n)} \|_{H^1_0(\Omega)}$ does. This is contained in the following estimate: There exists a two constants $c_\Omega, C_{\Omega,F}$ depending only on Ω and F such that:

$$I[u] \ \geqslant \ c_\Omega \| u \|^2_{H^1_0(\Omega)} - C_{\Omega,F} \ . \tag{7}$$

This is sometimes called *coercivity* of I. This is an immediate consequence of Young's inequality followed by the Poincare estimate. In particular, we have the pointwise bound $|Fw| \leqslant \epsilon w^2 + \epsilon^{-1}F^2$ for any $\epsilon > 0$. Thus:

$$|\iint_\Omega F w \, dx| \ \leqslant \ \epsilon \| w \|^2_{L^2(\Omega)} + \epsilon^{-1} \| F \|^2_{L^2(\Omega)} \ .$$

Then (7) follows by choosing ϵ so small that $\epsilon \ll C_\Omega$ where C_Ω is the constant in Poincare's inequality.

4. Two basic properties of Hilbert spaces

Recall that a sequence w_n in a Hilbert space $\mathcal{H}$ is said to *converge weakly* to a vector u if:

$$\lim_n \langle w_n, \varphi \rangle \ = \ \langle u, \varphi \rangle \ , \quad \text{for all } \varphi \in \mathcal{H} \ .$$

Notationaly we write $w_n \rightharpoonup u$. In terms of weak convergence we have the following result, whose proof is elementary, but nonetheless has far reaching consequences:

Proposition 4.1. *For any Hilbert space $\mathcal{H}$ the following two properties hold:*

i) If $w_n \rightharpoonup u$, then one has:

$$\| u \|_{\mathcal{H}} \ \leqslant \ \underline{\lim}_n \| w_n \|_{\mathcal{H}} \ . \tag{8}$$

ii) If w_n is any sequence with the uniform bound $\| w_n \|_{\mathcal{H}} \leqslant C$, then there exists a subsequence $w_{n_k} \rightharpoonup u$ for some $u \in \mathcal{H}$.

Proof. For the sake of completeness, we sketch a proof. To prove (8) simply write:

$$\| u - w_n \|_{\mathcal{H}}^2 \ = \ \| u \|_{\mathcal{H}}^2 - 2\langle u, w_n\rangle + \| w_n \|_{\mathcal{H}}^2 \ ,$$

and then take $\underline{\lim}_{\,n}$ of both sides.

To prove the weak sequential compactness, notice that a sequence converges weakly iff it converges strongly when projected to any finite dimensional subspace (this is essentially the definition of weak convergence). WLOG we may assume $\mathcal{H}$ is separable, for we can work on $\overline{\text{span}\{w_n\}}$. Then the previous requirement can be reduced to projected strong convergence in the nested collection of finite dimensional subspaces $\mathcal{K}_j = \text{span}\{e_1, \ldots, e_j\}$, where $\{e_i\}_{i=1}^{\infty}$ is some orthonormal basis of $\mathcal{H}$. Denote by P_j the corresponding projections. Then by using the compactness of closed bounded sets in each $\mathcal{K}_j$, and by repeatedly passing to subsequences whenever necessary, we can find a sequence of "overlapping" vectors $u_j \in \mathcal{H}$ with (strongly for each fixed j):

$$P_j w_{n_k} \ \to \ u_j \ , \quad \text{where } P_l u_j = u_l \text{ for } l \leqslant j \ .$$

Via the uniform bound on $P_j w_n$ we automatically have $\| u_j \|_{\mathcal{H}} \leqslant C$, and because we also have $P_l u_j = u_l$ when $l \leqslant j$ the sequence u_j is Cauchy. Setting u to be the corresponding limit we have the finite dimensional convergence $P_j w_{n_k} \to P_j u$, for each fixed j, as desired. ☐

5. The existence theorem

It is now a relatively simple matter to find a solution $v \in H_0^1(\Omega)$ to the problem (6). By the estimate (7) the functional $I[\cdot]$ is bounded from below on $H_0^1(\Omega)$, so it has a greatest lower bound I_0. Letting $w^{(n)}$ to be a minimizing sequence, we see again from (7) that $\| w^{(n)} \|_{H_0^1(\Omega)} \leqslant C$ for some universal constant C (depending on Ω and F). Then by ii) of Proposition 4.1 there exists a $v \in H_0^1(\Omega)$ and a minimizing subsequence with $w^{(n_k)} \rightharpoonup v$. By the lower semi-continuity estimate (8) and the definition of weak convergence, we have $I[v] \leqslant I_0$, so in fact $I[v] = I_0$ as desired.

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