SCHAUDER THEORY I: INTERIOR ESTIMATES FOR LAPLACE

JACOB STERBENZ

Abstract. These are companion notes for Math 231A on lectures from Wednesday 10/7/09 to Monday 10/12/09.

1. Weakly Harmonic Functions and Weyl's Theorem

Recall that a function $u \in L^1_{loc}$ is said to be *weakly harmonic* in a domain $\Omega \subseteq \mathbb{R}^n$ if:

$$\iint_\Omega u\Delta\varphi \, dx \ = \ 0 \ , \qquad \forall \ \varphi \in C_0^\infty(\Omega) \ .$$

Notice that if $u \in H^1(\Omega)$ then one can integrate by parts and the definition becomes:

$$\iint_\Omega \nabla u \cdot \nabla\varphi \, dx \ = \ 0 \ , \qquad \forall \ \varphi \in C_0^\infty(\Omega) \ .$$

The most basic fact about weakly harmonic functions in the following.

Theorem 1.1 (Weyl's Theorem)**.** *Let $u \in L^1_{loc}$ be weakly harmonic in Ω. Then by changing u on a set of measure zero we have $u \in C^\infty(\Omega)$, and for each subdomain $\Omega' \subset \Omega$ and integer $k \geqslant 0$, there is a constant $C = C(\Omega', \Omega, k)$ such that:*

$$\| \nabla^k u \|_{L^\infty(\Omega')} \ \leqslant \ C \| u \|_{L^1(\Omega)} \ . \tag{1}$$

Proof. The idea of the proof is to first establish (1) assuming that the solution u is *already smooth*. Then we can conclude the statement of the theorem assuming only that $u \in L^1_{loc}$ via the Arzelà-Ascoli theorem and a suitable approximation procedure.

The proof of (1) for smooth u is in Evans Chapter 2.2, but we also give a quick proof here. As usual, we induct on the value of k. If $k = 0$ then (1) follows immediately from the average value property over solid balls $B(x_0, r) \subseteq \Omega$ with $x \in \Omega'$. Notice that the constant C in (1) will become large via this procedure as $\partial\Omega'$ gets closer to $\partial\Omega$ (basically it behaves like $distance^{-n}$). Once we have (1) for $k = 0$, we can prove it for $k = 1$ by differentiating and then using the average value property and the divergence theorem:

$$\partial_k u(x_0) \ = \ \frac{1}{|B(x_0,r)|} \iint_{B(x_0,r)} \partial_k u \ dx \ = \ \frac{1}{|B(x_0,r)|} \int_{\partial B(x_0,r)} \frac{x_k}{|x|} u \ dS \ .$$

Using the L^∞ estimate for u just obtained on the RHS we arrive at the bound:

$$\| \nabla u \|_{L^\infty(\Omega')} \ \leqslant \ C \| u \|_{L^1(\Omega)} \ ,$$

for any nested pair $\Omega' \subset \Omega$. By repeating this k times for a sequence of such nested domains, we have by induction and Hölder's inequality the estimate (1).

It remains to deal with the general case $u \in L^1_{loc}$. Here we only need to show that for any such function which is also weakly harmonic in Ω, then for every smaller set $\Omega' \subset \Omega$ there exists a sequence of smooth functions u^ϵ, harmonic in Ω' for $0 < \epsilon \ll 1$ sufficiently small, and with $u^\epsilon \to u$ in $L^1(\Omega')$. This is provided via the standard mollification:

$$u^\epsilon(x) \ = \ \iint_{\mathbb{R}^n} \epsilon^{-n} \varphi_0\big(\epsilon^{-1}(x-y)\big) u(y) \ dy \ ,$$

where $\varphi_0 \geqslant 0$ is in C^∞_0 and $\int \varphi_0 dx = 1$. □

As an immediate consequence of estimate (1) we have the following:

Lemma 1.2 (Local Decay of Integrals for Harmonic Functions)**.** *Let* $\Delta u = 0$ *be in* $H^1(\Omega)$*. Then there exists a universal constant* $C > 0$ *such that for any pair of concentric balls* $B_\rho(x_0) \subseteq B_r(x_0) \subseteq \Omega$ *one has:*

$$\iint_{B_\rho(x_0)} |\nabla u|^2 dx \ \leqslant \ C \left(\frac{\rho}{r}\right)^n \iint_{B_r(x_0)} |\nabla u|^2 dx \ , \tag{2}$$

$$\iint_{B_\rho(x_0)} |\nabla u - \underline{\nabla u}_{x_0,\rho}|^2 dx \ \leqslant \ C \left(\frac{\rho}{r}\right)^{n+2} \iint_{B_r(x_0)} |\nabla u - \underline{\nabla u}_{x_0,r}|^2 dx \ , \tag{3}$$

where $\underline{f}_{x_0,r} = |B_r(x_0)|^{-1} \int_{B_r(x_0)} f \, dx$*.*

Proof. Notice that both of these bounds are invariant under the change of scale $u(x) \rightsquigarrow u(\lambda x)$ for any $\lambda > 0$. Thus, we may assume that $r = 1$. In both cases we may assume WLOG $\rho \leqslant \frac{1}{2}$ for otherwise the estimate is immediate.

The first bound (2) is direct consequence of (1). Using this estimate with $k = 0$, we have for any harmonic u the inequalities:

$$\| u \|_{L^2(B_\rho(x_0))} \lesssim \rho^{\frac{n}{2}} \| u \|_{L^\infty(B_\rho(x_0))} \leqslant \rho^{\frac{n}{2}} \| u \|_{L^1(B_1(x_0))} \lesssim \rho^{\frac{n}{2}} \| u \|_{L^2(B_1(x_0))} \ . \tag{4}$$

Applying this to the harmonic function ∇u instead of u yields the desired result.

Notice that by the same kind of reasoning used to show (4) we also have for any harmonic function in $B_1(x_0)$:

$$\| \nabla u \|_{L^2(B_\rho(x_0))} \ \leqslant \ C\rho^{\frac{n}{2}} \| u \|_{L^2(B_1(x_0))} \ .$$

We apply this to prove (3) by first using the Poincare estimate on the harmonic function $(\nabla u - \underline{\nabla u}_{x_0,\rho})$ (which has zero average), yielding the estimate:

$$\begin{aligned} \iint_{B_\rho(x_0)} |\nabla u - \underline{\nabla u}_{x_0,\rho}|^2 dx \lesssim \ & \rho^2 \iint_{B_\rho(x_0)} |\nabla(\nabla u - \underline{\nabla u}_{x_0,\rho})|^2 dx \ , \\ \lesssim \ & \rho^{n+2} \iint_{B_1(x_0)} |\nabla u - \underline{\nabla u}_{x_0,\rho}|^2 dx \ . \end{aligned}$$

We conclude by using the average value property which allows us to trade the averages via $\underline{\nabla u}_{x_0,\rho} = \underline{\nabla u}_{x_0,1}$. □

2. Local Approximation by Harmonic Functions

Our next order of business is to understand the regularity of weak solutions to Poisson's equation $\Delta w = F$. We will approach this subject in such a way that we assume as little as possible about F. As a first step in this direction, we pause to discuss how well arbitrary H^1 functions can be approximated by harmonic functions. This is contained in the following elementary result.

Lemma 2.1 (Harmonic Approximation Lemma)**.** *Let $w \in H^1\big(B_r(x_0)\big)$, and let u weakly solve the Dirichlet problem: $\Delta u = 0$ in $B_r(x_0)$ with $u|_{\partial B_r(x_0)} = w|_{\partial B_r(x_0)}$ (in the trace sense). Then there is a universal $C > 0$ such that for any $0 < \rho \leqslant r$ both:*

$$\begin{aligned} &\iint_{B_\rho(x_0)} |\nabla w|^2 dx \\ &\qquad \leqslant C\Big[\Big(\frac{\rho}{r}\Big)^n \iint_{B_r(x_0)} |\nabla w|^2 dx + \iint_{B_r(x_0)} |\nabla(u-w)|^2 dx \Big] \ , \end{aligned} \tag{5}$$

$$\begin{aligned} &\iint_{B_\rho(x_0)} |\nabla w - \underline{\nabla w}_{x_0,\rho}|^2 dx \\ &\qquad \leqslant C\Big[\Big(\frac{\rho}{r}\Big)^{n+2} \iint_{B_r(x_0)} |\nabla w - \underline{\nabla w}_{x_0,r}|^2 dx + \iint_{B_r(x_0)} |\nabla(u-w)|^2 dx \Big] \ , \end{aligned} \tag{6}$$

where $\underline{w}_{x_0,r} = |B_r(x_0)|^{-1} \int_{B_r(x_0)} w \, dx$.

Proof. To prove (5), we use the triangle inequality and estimate (2) to bound:

$$\begin{aligned} \iint_{B_\rho(x_0)} |\nabla w|^2 dx \ &\lesssim \ \iint_{B_\rho(x_0)} |\nabla u|^2 dx + \iint_{B_\rho(x_0)} |\nabla(u-w)|^2 dx \ , \\ &\lesssim \ \Big(\frac{\rho}{r}\Big)^n \iint_{B_r(x_0)} |\nabla u|^2 dx + \iint_{B_r(x_0)} |\nabla(u-w)|^2 dx \ , \\ &\lesssim \Big(\frac{\rho}{r}\Big)^n \iint_{B_r(x_0)} |\nabla w|^2 dx + \iint_{B_r(x_0)} |\nabla(u-w)|^2 dx \ . \end{aligned}$$

To prove (6) we begin with:

$$\begin{aligned} \iint_{B_\rho(x_0)} |\nabla w - \underline{\nabla w}_{x_0,\rho}|^2 dx \ \lesssim \ &\iint_{B_\rho(x_0)} |\nabla u - \underline{\nabla u}_{x_0,\rho}|^2 dx + \iint_{B_\rho(x_0)} |\nabla(u-w)|^2 dx \\ &+ \iint_{B_\rho(x_0)} |\underline{\nabla(u-w)}_{x_0,\rho}|^2 dx \ . \end{aligned}$$

Using again the estimate $\| \, \underline{f} \, \|_{L^2} \leqslant \| \, f \, \|_{L^2}$ to treat the last term on the RHS above, and estimate (3) to treat the first, we have:

$$\begin{aligned} \iint_{B_\rho(x_0)} |\nabla w - \underline{\nabla w}_{x_0,\rho}|^2 dx \ &\lesssim \ \Big(\frac{\rho}{r}\Big)^{n+2} \iint_{B_r(x_0)} |\nabla u - \underline{\nabla u}_{x_0,r}|^2 dx \\ &\qquad + \iint_{B_r(x_0)} |\nabla(u-w)|^2 dx \ , \\ &\lesssim \ \Big(\frac{\rho}{r}\Big)^{n+2} \iint_{B_r(x_0)} |\nabla w - \underline{\nabla w}_{x_0,r}|^2 dx \\ &\qquad + \iint_{B_r(x_0)} |\nabla(u-w)|^2 dx \ , \end{aligned}$$

where we have bounded the expression $\iint_{B_r(x_0)}|\underline{\nabla u}_{x_0,r} - \underline{\nabla w}_{x_0,r}|^2 dx$ which comes up in the second inequality with yet another use of $\| \underline{f} \|_{L^2} \leqslant \| f \|_{L^2}$. $\square$

3. Local Decay of Integrals of Solutions to the Poisson Equation

We would now like to prove decay estimates line (2)–(3) for solutions to the equation $\Delta w = F$. Recall that we say this equations holds *weakly* in a domain Ω for a $u \in H^1(\Omega)$ if:

$$\iint_\Omega \nabla w \cdot \nabla \varphi \, dx \ + \iint_\Omega F \varphi \, dx \ = \ 0 \ , \qquad \forall \ \varphi \in C_0^\infty(\Omega) \ . \tag{7}$$

The basic result of this section is the following:

Proposition 3.1 (Decay of Local Integrals for Poisson's Equation)**.** *Suppose that* $w \in H^1(\Omega)$ *solves* (7)*. Then if* $F = f + \sum_i \partial_i f_i$ *(weakly) where* $f \in L^p(\Omega)$ *and* $f_i \in L^q(\Omega)$ *for* $\alpha = 2 - \frac{n}{p} = 1 - \frac{n}{q} \in (0,1)$*, then for any set of concentric balls* $B_\rho(x_0) \subseteq B_r(x_0) \subseteq \Omega$ *the following estimate holds:*

$$\iint_{B_\rho(x_0)} |\nabla w|^2 dx \lesssim \left(\frac{\rho}{r}\right)^{n-2+2\alpha} \left(\| f \|_{L^p(\Omega)}^2 + \sum_i \| f_i \|_{L^q(\Omega)}^2 + \| w \|_{H^1(\Omega)}^2 \right) \ , \tag{8}$$

Furthermore, if we instead assume that $\alpha = 1 - \frac{n}{p} \in (0,1)$ *and* $f_i \in C^\alpha(\Omega)$*, then the following improvement is valid:*

$$\iint_{B_\rho(x_0)} |\nabla w - \underline{\nabla w}_{x_0,\rho}|^2 dx \lesssim \left(\frac{\rho}{r}\right)^{n+2\alpha} \left(\| f \|_{L^p(\Omega)}^2 + \sum_i \| f_i \|_{C^\alpha(\Omega)}^2 + \| w \|_{H^1(\Omega)}^2 \right) \ . \tag{9}$$

Before embarking on the proof, we need one piece of technology. Recall the following:

Theorem 3.2 (Sobolev Embedding Theorem)**.** *For any test function* $\varphi \in C_0^\infty(\Omega)$ *the following estimate is valid* $1 \leqslant p < n$*:*

$$\| \varphi \|_{L^{\frac{np}{n-p}}(\Omega)} \ \leqslant C_p \| \nabla \varphi \|_{L^p(\Omega)} \ . \tag{10}$$

Proof. The proof of this theorem is given in Theorem 1 of Section 5.6 of Evans. See also Section 8.3 of [1]. $\square$

Proof of estimate (8)*.* To simplify notation, we drop the x_0 index from $B_r(x_0)$ etc. Using the harmonic approximation estimate (5) we immediately have that:

$$\iint_{B_\rho} |\nabla w|^2 dx \lesssim \left(\frac{\rho}{r}\right)^n \iint_{B_r} |\nabla w|^2 dx + \iint_{B_r} |\nabla (w-u)|^2 dx \ , \tag{11}$$

where $\Delta u = 0$ in B_r with $(w-u) \in H_0^1(B_r)$. The first term on the RHS above is of the desired form, so it remains to estimate the second. Via integration by parts and Hölder's inequality we have:

$$\begin{aligned} &\iint_{B_r} |\nabla(w-u)|^2 dx \ , \\ = \ & -\iint_{B_r} (w-u)\Delta(w-u) \ dx \ , \\ = \ & -\iint_{B_r} (w-u) f \ dx + \iint_{B_r} \nabla(w-u)\cdot(f_i) \ dx \ , \\ \leqslant \ & \| \, w-u \, \|_{L^{\frac{2n}{n-2}}(B_r)} \| \, f \, \|_{L^{\frac{2n}{n+2}}(B_r)} + \| \, \nabla(w-u) \, \|_{L^2(B_r)} \cdot \| \, (f_i) \, \|_{L^2(B_r)} \ . \end{aligned}$$

Therefore, by applying the Sobolev estimate (10) to the first term on the RHS above, and then Young's inequality to both terms, we arrive at the bound:

$$\iint_{B_r} |\nabla(w-u)|^2 dx \ \lesssim \ \epsilon \iint_{B_r} |\nabla(w-u)|^2 dx + \epsilon^{-1} \left(\| \, f \, \|^2_{L^{\frac{2n}{n+2}}(B_r)} + \| \, (f_i) \, \|^2_{L^2(B_r)} \right) \ .$$

Absorbing the first term on the RHS to the left, and then again using various instances of Hölders inequality we have:

$$\iint_{B_r} |\nabla(w-u)|^2 dx \ \lesssim \ r^{n-2+2\alpha} \left(\| \, f \, \|^2_{L^p(B_r)} + \| \, (f_i) \, \|^2_{L^q(B_r)} \right) \ .$$

Plugging this last line into the RHS of (11) we have:

$$\| \, \nabla w \, \|^2_{L^2(B_\rho)} \ \lesssim \ \left(\frac{\rho}{r}\right)^n \| \, \nabla w \, \|^2_{L^2(B_r)} + r^{n-2+2\alpha} \left(\| \, f \, \|^2_{L^p(B_r)} + \| \, (f_i) \, \|^2_{L^q(B_r)} \right) \ .$$

Unfortunately, this is not quite the same form as (8). However, this is fixed using the following lemma. □

Lemma 3.3 (Iteration Lemma)**.** *Let $I(\rho)$ be a monotonically increasing quantity defined for $0 < \rho \leqslant 1$ and such that:*

$$I(\rho) \ \leqslant \ A\left(\frac{\rho}{r}\right)^a I(r) + r^b N \ , \tag{12}$$

for any $\rho \leqslant r \leqslant 1$ and fixed $A, N > 0$ and $0 < b < a$. Then there exists a universal constant $C = C(A, a, b)$ such that:

$$I(\rho) \ \leqslant \ C\left(\frac{\rho}{r}\right)^b \Big[I(r) + N \Big] \ , \qquad \text{when } 0 < \rho \leqslant r \leqslant 1 \ . \tag{13}$$

Proof. Fix r. By monotonicity, it suffices to prove this for all ρ of the form $\rho_k = M^{-k} r$, where M is a sufficiently large parameter to be chosen in a moment. Set $I_j = I(\rho_j)$ where $0 \leqslant j \leqslant k$. This $I_0 = I(r)$ and $I_k = I(\rho)$. We are trying to show that:

$$I_k \ \leqslant C M^{-kb}(I_0 + N) \ . \tag{14}$$

From (12) we have the one step bound:

$$I_j \ \leqslant \ A M^{-a} I_{j-1} + (M^{1-j} r)^b N \ .$$

By inductively repeating this estimate ℓ times starting at $j = \ell$ we have the series:

$$I_\ell \ \leqslant \ (A M^{-a})^\ell I_0 + \sum_{j=1}^{\ell} (A M^{-a})^{j-1} (M^{j-\ell} r)^b N \ .$$

Using this with $k = \ell$, and choosing M so large that $AM^{b-a} \leqslant \frac{1}{2}$ we have:

$$I_k \ \leqslant \ M^{-bk} I_0 + r^b A^{-1} M^a \sum_{j=1}^{k} 2^{-j} \cdot M^{-bk} N \ .$$

The dyadic sum converges, and using $r \leqslant 1$ we have (14). □

Proof of estimate (9). This is done in almost exactly the same was as above. Starting with the approximation bound (6) we have:

$$\iint_{B_\rho} |\nabla w - \underline{\nabla w}_{x_0,\rho}|^2 dx \lesssim \left(\frac{\rho}{r}\right)^n \iint_{B_r} |\nabla w - \underline{\nabla w}_{x_0,\rho}|^2 dx + \iint_{B_r} |\nabla(w-u)|^2 dx \ , \tag{15}$$

where again $\Delta u = 0$ in B_r with $(w-u) \in H^1_0(B_r)$. We begin to estimate the second term on the RHS as above, which gives the preliminary estimate:

$$\begin{aligned} \| \nabla w - \underline{\nabla w}_{x_0,\rho} \|^2_{L^2(B_\rho)} \ &\lesssim \ \left(\frac{\rho}{r}\right)^{n+2} \| \nabla w - \underline{\nabla w}_{x_0,\rho} \|^2_{L^2(B_r)} \\ &\qquad + \| f \|^2_{L^{\frac{2n}{n+2}}(B_r)} + \| (f_i - f_i(x_0)) \|^2_{L^2(B_r)} \ . \end{aligned}$$

Here we have subtracted off the center values of the f_i in the corresponding integral:

$$\begin{aligned} \iint_{B_r} (w-u) \nabla \cdot (f_i) \ dx \ &= \ \iint_{B_r} (w-u) \nabla \cdot (f_i(x) - f_i(x_0)) \ dx \ , \\ &= \ - \iint_{B_r} \nabla(w-u) \cdot (f_i(x) - f_i(x_0)) \ dx \ . \end{aligned}$$

We now wish to estimate the last two terms on the RHS of the previous estimate in such a way that we can apply the iteration estimate (13). For the first such term, we directly have by an application of Hölder's inequality and the definition $\alpha = 1 - \frac{n}{p}$ that:

$$\| f \|_{L^{\frac{2n}{n+2}}(B_r)} \ \leqslant \ r^{\frac{n}{2}+\alpha} \| f \|_{L^p(B_r)} \ .$$

To conclude, we claim that there exists a universal constant C depending only on α such that for all $f_i \in C^\alpha(B_r(x_0))$ the following estimate holds (uniform in r):

$$\| f_i - f_i(x_0) \|_{L^2(B_r(x_0))} \ \leqslant \ C r^{\frac{n}{2}+\alpha} \| f_i \|_{C^\alpha(B_r(x_0))} \ . \tag{16}$$

Indeed, WLOG we can assume that $f_i(x_0) = 0$ because such an estimate is clearly true for constant functions. Then by the definition of C^α we have:

$$|f_i(x - x_0)| \ \leqslant \ |x - x_0|^\alpha \| f_i \|_{C^\alpha(B_r(x_0))} \ ,$$

so integrating this inequality squared we have for f_i vanishing at x_0:

$$\| f_i \|^2_{L^2(B_r)} \ \leqslant \ \| f_i \|^2_{C^\alpha(B_r)} \iint_{|x| \leqslant r} |x|^{2\alpha} dx \ \leqslant \ C_\alpha r^{n+2\alpha} \| f_i \|^2_{C^\alpha(B_r)} \ .$$

This concludes the demonstration of (9). □

4. Campanato's Inequality

In the proof of the second estimate (9) in the last section, we used the fact that any function in C^α has a decaying L^2 norm on small balls if we first subtract off its central value (estimate (16)). We would like to show a kind of converse to this estimate, which will allow us to turn the integral decay estimates (8)–(9) into Hölder regularity. This is done via the following:

Theorem 4.1 (Campanato's Inequality)**.** *Let $\Omega \subset\subset \mathbb{R}^n$ be an an open set with Lipschitz boundary $\partial\Omega$. Then if $w \in L^2(\Omega)$ is a function such that the following estimate holds uniformly on all balls $B_\rho(x_0)$ with $x_0 \in \Omega$ and $0 < \rho \leqslant 1$:*

$$\| w - \underline{w}_{x_0,\rho} \|_{L^2(B_\rho(x_0)\cap\Omega)} \ \leqslant\ \rho^{\frac{n}{2}+\alpha} N \ , \tag{17}$$

for some constant $N > 0$ and $\alpha \in (0,1)$. Here we are defining the averages over truncated balls:

$$\underline{w}_{x_0,\rho} \ :=\ \frac{1}{|B_\rho(x_0)\cap\Omega|}\iint_{B_\rho(x_0)\cap\Omega} w(x)dx \ .$$

Then by changing w on a set of measure zero we have $w \in C^\alpha(\overline{\Omega})$, and there exists a constant $C = C(\alpha, \Omega) > 0$ such that:

$$\| w \|_{C^\alpha(\overline{\Omega})} \ \leqslant\ C(N + \| w \|_{L^2(\Omega)}) \ . \tag{18}$$

Proof. Since the estimate is linear in w, we may assume that $N = 1$. Recall that the definition of the Hölder norm is:

$$\| w \|_{C^\alpha(\overline{\Omega})} \ :=\ \sup_{x\in\Omega} |w(x)| + \sup_{x,y\in\Omega} \frac{|w(x) - w(y)|}{|x - y|^\alpha} \ .$$

We'll bound first the L^∞ norm and then deal with the Hölder component.

The Lipschitz requirement on $\partial\Omega$ comes in as follows: It is not too difficult to see that for any such domain there exists a constant $c = c(\Omega) > 0$ such that:

$$|B_\rho(x) \cap \Omega| \ \geqslant\ c\rho^n \ ,$$

uniform in $x \in \Omega$ and $\rho \leqslant 1$. In particular, if $x, y \in \Omega$ are any two points, and $1 \geqslant \rho, \sigma \geqslant 2|x - y|$ are any two real numbers with $\frac{1}{10}\sigma \leqslant \rho \leqslant 10\sigma$, then we have the uniform inequalities (for sufficiently small c):

$$|B_\rho(x) \cap B_\sigma(y) \cap \Omega| \ \geqslant\ c|B_\rho(x) \cap \Omega| \ \geqslant\ c^2\rho^n \ , \tag{19}$$

$$|B_\rho(x) \cap B_\sigma(y) \cap \Omega| \ \geqslant\ c|B_\sigma(y) \cap \Omega| \ \geqslant\ c^2\sigma^n \ . \tag{20}$$

Now fix $x, y \in \Omega$ and $\rho, \sigma \geqslant 2|x - y|$ such that $B_\rho(x), B_\sigma(y) \subseteq \Omega$ and $\frac{1}{10}\sigma \leqslant \rho \leqslant 10\sigma$. Then we study the differences:

$$|\underline{w}_{x,\rho} - \underline{w}_{y,\sigma}| \ \leqslant\ |w(z) - \underline{w}_{x,\rho}| + |w(z) - \underline{w}_{y,\sigma}| \ .$$

Averaging in z both sides of this over the intersection $\mathcal{B} = B_\rho(x) \cap B_\sigma(y) \cap \Omega$, using (19)–(20), the Cauchy Schwartz inequality and also the similarity bounds on ρ, σ

we have:

$$
\begin{aligned}
&|\underline{w}_{x,\rho} - \underline{w}_{y,\sigma}| \ , \\
&\leqslant \frac{1}{|\mathcal{B}|}\iint_{\mathcal{B}} |w(z) - \underline{w}_{x,\rho}| \ dz + \frac{1}{|\mathcal{B}|}\iint_{\mathcal{B}} |w(z) - \underline{w}_{y,\sigma}| \ dz \ , \\
&\leqslant C\left(\frac{1}{|B_\rho(x)\cap\Omega|}\iint_{B_\rho(x)} |w(z) - \underline{w}_{x,\rho}| \ dz + \frac{1}{|B_\sigma(y)\cap\Omega|}\iint_{B_\sigma(y)} |w(z) - \underline{w}_{y,\sigma}| \ dz\right) \ , \\
&\leqslant C\left(\rho^{-\frac{n}{2}} \| w - \underline{w}_{x,\rho} \|_{B_\rho(x)} + \sigma^{-\frac{n}{2}} \| w - \underline{w}_{y,\sigma} \|_{B_\sigma(y)}\right) \ , \\
&\leqslant C\rho^\alpha \ = \ \widetilde{C}\sigma^\alpha \ .
\end{aligned} \tag{21}
$$

If we set $x = y$ and $\rho = r, \sigma = \frac{1}{2}r$ in the above estimate we have:

$$
|\underline{w}_{x,r} - \underline{w}_{x,\frac{1}{2}r}| \ \leqslant \ Cr^\alpha \ ,
$$

for any nested sequence of balls $B_{\frac{1}{2}r} \subset B_r(x)$. Telescoping this estimate and summing dyadically over $j \leqslant \ell \leqslant k-1$, we have:

$$
|\underline{w}_{x,2^{-j}r} - \underline{w}_{x,2^{-k}r}| \leqslant C\sum_{\ell=j}^{k-1} 2^{-\alpha\ell} r^\alpha \ \leqslant \ C_\alpha(2^{-j}r)^\alpha \ . \tag{22}
$$

In particular, the averages $w^{(j)}(x) = \underline{w}_{x,2^{-j}}$ form a $C^0(\overline{\Omega})$ Cauchy sequence which converges uniformly to a continuous limit $\widetilde{w}$. By the Lebesgue differentiation theorem, this limit function is such that $\widetilde{w} = w$ almost everywhere. In particular, by changing w on a set of measure zero it becomes continuous, and for any point $x \in \Omega$ we have from taking $k \to \infty$ and $r = 1$ in (22), followed by a use of the Cauchy-Schwartz inequality, the uniform L^∞ bound:

$$
|w(x)| \ \leqslant \ C(1 + \| w \|_{L^2(B_1(x))}) \ .
$$

Thus, we have control of the sup norm component of $\| w \|_{C^\alpha(\overline{\Omega})}$.

It remains to bound the difference quotient portion of the Hölder norm. For $x, y \in \Omega$ we set $\rho = \sigma = 2|x-y|$ in estimate (21) and use the $k \to \infty$ version of (22), which gives:

$$
\begin{aligned}
|w(x) - w(y)| \ &\leqslant \ |w(x) - \underline{w}_{x,\rho}| + |w(y) - \underline{w}_{y,\rho}| + |\underline{w}_{x,\rho} - \underline{w}_{y,\rho}| \ , \\
&\leqslant \ C\rho^\alpha \ = \ C|x-y|^\alpha \ .
\end{aligned}
$$

This is enough to conclude our demonstration of (18). □

5. Interior Schauder Estimates for the Laplacian

We would like to use the estimates (8)–(9) proved in the above section to show that any weak solution to $\Delta w = F$ has a certain amount of Hölder regularity assuming that F does. For example, *assuming* that $\nabla^2 w \in C^\alpha(\Omega)$ implies that $F \in C^\alpha(\Omega)$, so in general we cannot do any better that gaining two derivatives over F. The following estimate shows that we in fact can recover this optimal amount of regularity:

Theorem 5.1 (Interior Schauder Estimates)**.** *Let $w \in H^1(\Omega)$ be a weak solution to (7). Then for every $\alpha \in (0,1)$ there exists a uniform constant $C = C(\alpha, \Omega)$ such that if $\Omega' \subset\subset \Omega$ is any smooth subdomain with $d = d(\Omega, \Omega') = dist(\partial\Omega, \partial\Omega')$ then the following estimates hold:*

i) If $F = f + \sum_i \partial_i f_i$ *(weakly) where* $f \in L^p(\Omega)$ *and* $f_i \in L^q(\Omega)$ *for* $\alpha = 2 - \frac{n}{p} = 1 - \frac{n}{q} \in (0,1)$*, then* $w \in C^\alpha(\Omega)$ *and:*

$$\| w \|_{C^\alpha(\Omega')} \leqslant Cd^{-\frac{n}{2}+1-\alpha} \big(\| f \|_{L^p(\Omega)} + \| (f_i) \|_{L^q(\Omega)} + \| w \|_{H^1(\Omega)} \big) \ . \tag{23}$$

ii) If instead assume that $f \in L^p(\Omega)$ *for* $\alpha = 1 - \frac{n}{p} \in (0,1)$*, and* $f_i \in C^\alpha(\Omega)$*, then* $\nabla w \in C^\alpha(\Omega)$ *and:*

$$\| \nabla w \|_{C^\alpha(\Omega')} \leqslant Cd^{-\frac{n}{2}-\alpha} \big(\| f \|_{L^p(\Omega)} + \| (f_i) \|_{C^\alpha(\Omega)} + \| w \|_{H^1(\Omega)} \big) \ . \tag{24}$$

As an immediate application of these bounds, we have the classical Schauder estimate:

Corollary 5.2 (Basic Interior $C^{2,\alpha}$ Estimate)**.** *Let* $\Delta w = F$ *in the sense of distributions. Then if* $F \in C^\alpha(\Omega)$ *with* $\alpha \in (0,1)$ *we must have* $w \in C^{2,\alpha}(\Omega)$*, we have for every subdomain* $\Omega' \subset\subset \Omega$ *a uniform constant* $C = C(\alpha, \Omega, \Omega') = C(\alpha, \Omega) dist^{-\frac{n}{2}-1-\alpha}(\partial\Omega, \partial\Omega')$ *such that:*

$$\| w \|_{C^{2,\alpha}(\Omega')} \leqslant C \big(\| F \|_{C^\alpha(\Omega)} + \| w \|_{H^1(\Omega)} \big) \ . \tag{25}$$

Proof of estimate (25)*.* By density, we can assume that $u \in C^\infty(\Omega)$. From the first estimate (23) and using the boundedness of F and Hölder's inequality, we have (25) for with the LHS replaced by $\| w \|_{C^\alpha(\Omega')}$ for $\alpha \in (0,1)$. Applying the second bound (24) and again using an L^∞ estimate for F, we can upgrade this to LHS=$\| w \|_{C^{1,\alpha}(\Omega')}$. Finally, by differentiating the equation $\Delta u = F$ and using the C^α estimate contained in (24) for the corresponding RHS, we have the full estimate (25). □

Proof of the estimates (23)–(24)*.* These estimates follow almost immediately from (8)–(9) combined with the inequality (17)–(18). For (24) this is direct. In the case of (23) we first use the (rescaled) Poincare estimate:

$$\| w - \underline{w}_{x_0,\rho} \|_{B_\rho(x_0)} \lesssim \rho \| \nabla w \|_{B_\rho(x_0)} \ ,$$

and then plug the LHS of this last line in the RHS of (23) to get the needed decay of the form (17). □

References

[1] Lieb, Elliott H.; Loss, Michael **Analysis.** Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001. xxii+346 pp.

Department of Mathematics, University of California, San Diego (UCSD) 9500 Gilman Drive, Dept 0112 La Jolla, CA 92093-0112 USA
E-mail address: `jsterben@math.ucsd.edu`